Lecture 3

- More on Maxwell
- Wave Equations
- Boundary Conditions
- Poynting Vector
- Transmission Line

Maxwell’s equations in differential form

$$\nabla.\mathbf{D} = \rho \qquad \textit{Gauss' law for electrostatics}$$

$$\nabla.\mathbf{B} = 0 \qquad \textit{Gauss' law for magnetostatics}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{dt} \qquad \textit{Ampere's law}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{dt} \qquad \textit{Faraday's law}$$

$$\nabla.\mathbf{J} = -\frac{\partial \rho}{\partial t} \qquad \textit{Equation of continuity}$$

$$\mathbf{D} = \varepsilon \mathbf{E}$$

$$\mathbf{B} = \mu \mathbf{H}$$

- Varying E and H fields are coupled

Electromagnetic waves in lossless media - Maxwell's equations

Maxwell

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{dt}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{dt}$$

$$\nabla . \mathbf{D} = \rho$$

$$\nabla . \mathbf{B} = 0$$

Equation of continuity

$$\nabla . \mathbf{J} = -\frac{\partial \rho}{\partial t}$$

Constitutive relations

$$\mathbf{D} = \varepsilon \mathbf{E} = \varepsilon_r \varepsilon_o \mathbf{E}$$

$$\mathbf{B} = \mu \mathbf{H} = \mu_r \mu_o \mathbf{H}$$

$$\mathbf{J} = \sigma \mathbf{E}$$

SI Units

- J Amp/ metre^2
- D Coulomb/metre^2
- H Amps/metre
- B Tesla
Weber/metre^2
Volt-Second/metre^2
- E Volt/metre
- ε Farad/metre
- μ Henry/metre
- σ Siemen/metre

Wave equations in free space

- In free space
  - $\sigma=0 \Rightarrow \mathbf{J}=0$
  - Hence: $$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{dt} = \frac{\partial \mathbf{D}}{dt}$$
    $$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{dt}}$$
  - Taking curl of both sides of latter equation:
    $$\nabla \times \nabla \times \mathbf{E} = -\nabla \times \frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial}{\partial t} \nabla \times \mathbf{B} = -\mu_o \frac{\partial}{\partial t} \nabla \times \mathbf{H}$$
    $$= -\mu_o \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{D}}{\partial t}\right)$$
    $$\boxed{\nabla \times \nabla \times \mathbf{E} = -\mu_o \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}}$$

Wave equations in free space cont.

$$\nabla \times \nabla \times \mathbf{E} = -\mu_o \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

- It has been shown (last week) that for any vector **A**

$$\nabla \times \nabla \times \mathbf{A} = \nabla \nabla . \mathbf{A} - \nabla^2 \mathbf{A}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the *Laplacian* operator

Thus:

$$\nabla \nabla . \mathbf{E} - \nabla^2 \mathbf{E} = -\mu_o \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

- There are no free charges in free space so $\nabla . \mathbf{E} = \rho = 0$ and we get

$$\boxed{\nabla^2 \mathbf{E} = \mu_o \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}}$$

A three dimensional wave equation

Wave equations in free space cont.

- Both **E** and **H** obey second order partial differential wave equations:

$$\nabla^2 \mathbf{E} = \mu_o \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$\nabla^2 \mathbf{H} = \mu_o \varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2}$$

- What does this mean
  - dimensional analysis ?

$$\frac{\text{Volts/metre}}{\text{metre}^2} = \mu_o \varepsilon \frac{\text{Volts/metre}}{\text{seconds}^2}$$

  - $\mu_o \varepsilon$ has units of velocity^{-2}
  - Why is this a wave with velocity $1/\sqrt{\mu_o \varepsilon}$?

Uniform plane waves - transverse relation of E and H

- Consider a uniform plane wave, propagating in the z direction. **E** is independent of x and y

$$\frac{\partial \mathbf{E}}{\partial x} = 0 \qquad \frac{\partial \mathbf{E}}{\partial y} = 0$$

In a source free region, $\nabla.\mathbf{D}=\rho=0$ (Gauss' law) :

$$\nabla.\mathbf{E} = \frac{\partial \mathbf{E}_x}{\partial x} + \frac{\partial \mathbf{E}_y}{\partial y} + \frac{\partial \mathbf{E}_z}{\partial z} = 0$$

E is independent of x and y, so

$$\frac{\partial E_x}{\partial x} = 0, \quad \frac{\partial E_y}{\partial y} = 0 \quad \Rightarrow \quad \frac{\partial E_z}{\partial z} = 0 \quad \boxed{\Rightarrow E_z = 0} \quad (E_z = \text{const is not a wave})$$

- So for a plane wave, **E** has no component in the direction of propagation. Similarly for **H.**
- Plane waves have only transverse **E** and **H** components.

Orthogonal relationship between E and H:

- For a plane z-directed wave there are no variations along *x* and *y*:

$$\nabla \times \mathbf{H} = -\mathbf{a}_x \frac{\partial H_y}{\partial z} + \mathbf{a}_y \frac{\partial H_x}{\partial z}$$

$$= \frac{\partial \mathbf{D}}{\partial t}$$

$$= \varepsilon \left(\mathbf{a}_x \frac{\partial E_x}{\partial t} + \mathbf{a}_y \frac{\partial E_y}{\partial t} + \cancel{\mathbf{a}_y \frac{\partial E_z}{\partial t}} \right)$$

$$\nabla \times A = \mathbf{a}_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \mathbf{a}_y \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \mathbf{a}_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

$$\nabla \times \mathbf{H} = \cancel{\mathbf{J}} + \frac{\partial \mathbf{D}}{dt}$$

- Equating terms:

$$-\frac{\partial H_y}{\partial z} = \varepsilon \frac{\partial E_x}{\partial t}$$

$$\frac{\partial H_x}{\partial z} = \varepsilon \frac{\partial E_y}{\partial t}$$

- and likewise for $\nabla \times \mathbf{E} = -\mu_o \, \partial \mathbf{H} / \partial t$:

$$\frac{\partial E_y}{\partial z} = \mu_o \frac{\partial H_x}{\partial t}$$

$$\frac{\partial E_x}{\partial z} = \mu_o \frac{\partial H_y}{\partial t}$$

- **Spatial rate of change of H is proportionate to the temporal rate of change of the orthogonal component of *E* & v.v. *at the same point in space***

Orthogonal and phase relationship between E and H:

- Consider a linearly polarised wave that has a transverse component in (say) the y direction only:

$$-\frac{\partial H_y}{\partial z} = \varepsilon \frac{\partial E_x}{\partial t}$$

$$\frac{\partial H_x}{\partial z} = \varepsilon \frac{\partial E_y}{\partial t}$$

$$E_y = E_o f(z - vt)$$

$$\Rightarrow \varepsilon \frac{\partial E_y}{\partial t} = -\varepsilon v E_o f'(z - vt) = \frac{\partial H_x}{\partial z}$$

$$\Rightarrow \quad H_x = -\varepsilon v E_o \int f'(z - vt) dz + const = -\varepsilon v E_o f(z - vt)$$

$$= -\varepsilon v E_y$$

$$H_x = -\sqrt{\frac{\varepsilon}{\mu_o}} E_y$$

- Similarly

$$\frac{\partial E_y}{\partial z} = \mu_o \frac{\partial H_x}{\partial t}$$

$$\frac{\partial E_x}{\partial z} = \mu_o \frac{\partial H_y}{\partial t}$$

$$H_y = \sqrt{\frac{\varepsilon}{\mu_o}} E_x$$

- H and E are in phase and orthogonal

$$H_x = -\sqrt{\frac{\varepsilon}{\mu_o}}E_y \qquad H_y = \sqrt{\frac{\varepsilon}{\mu_o}}E_x$$

- The ratio of the magnetic to electric fields strengths is:

$$\frac{\sqrt{E_x^2 + E_y^2}}{\sqrt{H_x^2 + H_y^2}} = \frac{E}{H} = \sqrt{\frac{\mu_o}{\varepsilon}} = \eta$$

Note:

$$\frac{E}{B} = \frac{E}{\mu_o H} = \frac{1}{\sqrt{\mu_o \varepsilon_o}} = c$$

which has units of impedance

$$\frac{Volts/metre}{amps/metre} = \Omega$$

- and the *impedance of free space* is:

$$\sqrt{\frac{\mu_o}{\varepsilon_o}} = \sqrt{\frac{4\pi \times 10^{-7}}{\frac{1}{36\pi} \times 10^{-9}}} = 120\pi = 377\Omega$$

Orientation of **E** and **H**

- For any medium the intrinsic impedance is denoted by η

$$\eta = -\frac{E_y}{H_x} = \frac{E_x}{H_y}$$

and taking the scalar product

$$\mathbf{E}.\mathbf{H} = E_x H_x + E_y H_y$$
$$= \eta H_y H_x - \eta H_x H_y = 0$$

so **E** and **H** are mutually orthogonal

- Taking the cross product of **E** and **H** we get the direction of wave propagation

$$\mathbf{E} \times \mathbf{H} = \mathbf{a}_z \left(E_x H_y - E_y H_x \right)$$
$$= \mathbf{a}_z \left(\eta H_y^2 - \eta H_x^2 \right)$$

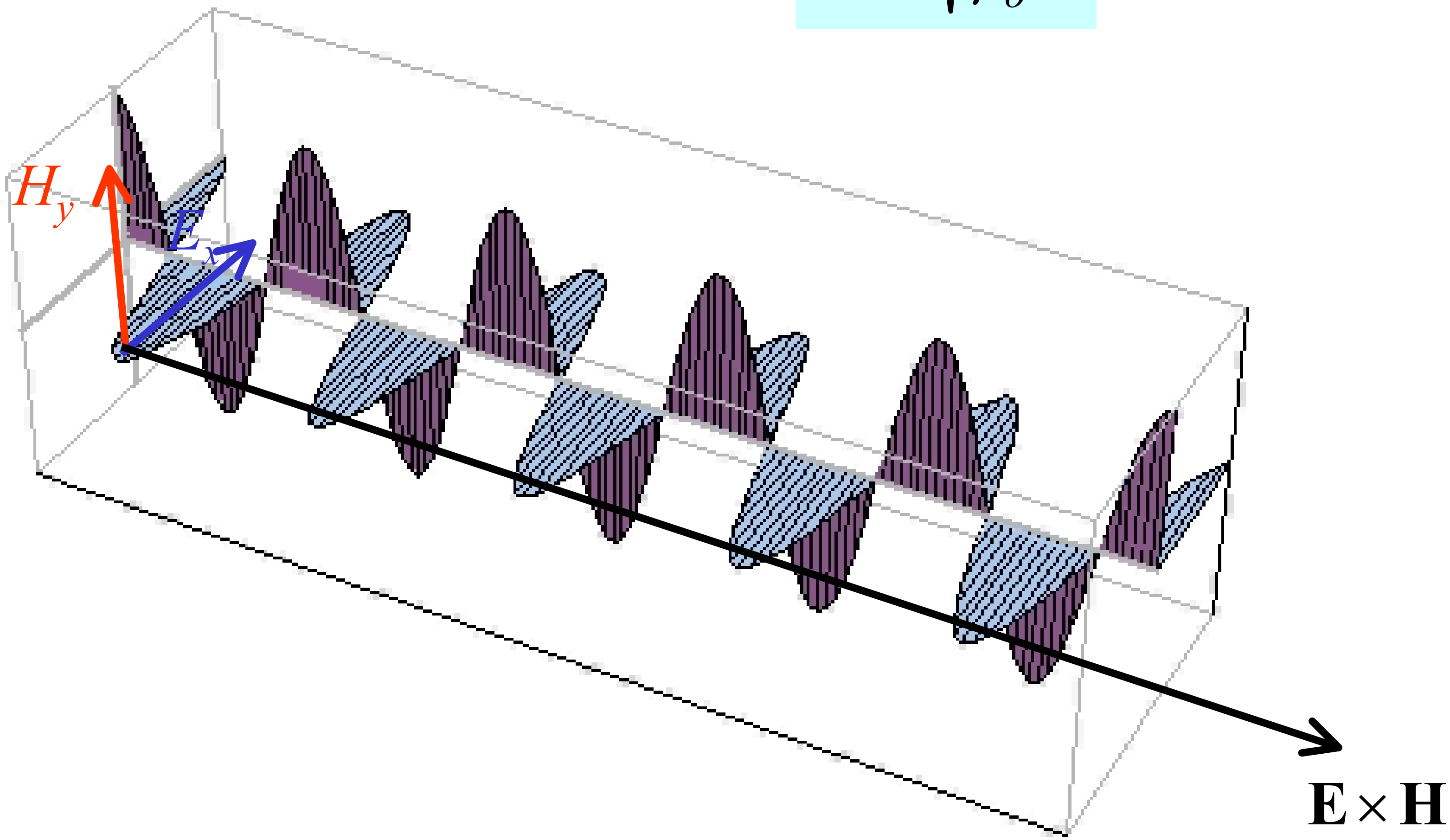
$$\mathbf{E} \times \mathbf{H} = \mathbf{a}_z \eta H^2$$

$$\mathbf{A} \times \mathbf{B} = \mathbf{a}_x \left(A_y B_z - A_z B_y \right) + \mathbf{a}_y \left(A_z B_x - A_x B_z \right) + \mathbf{a}_z \left(A_x B_y - A_y B_x \right)$$

A ‘horizontally’ polarised wave

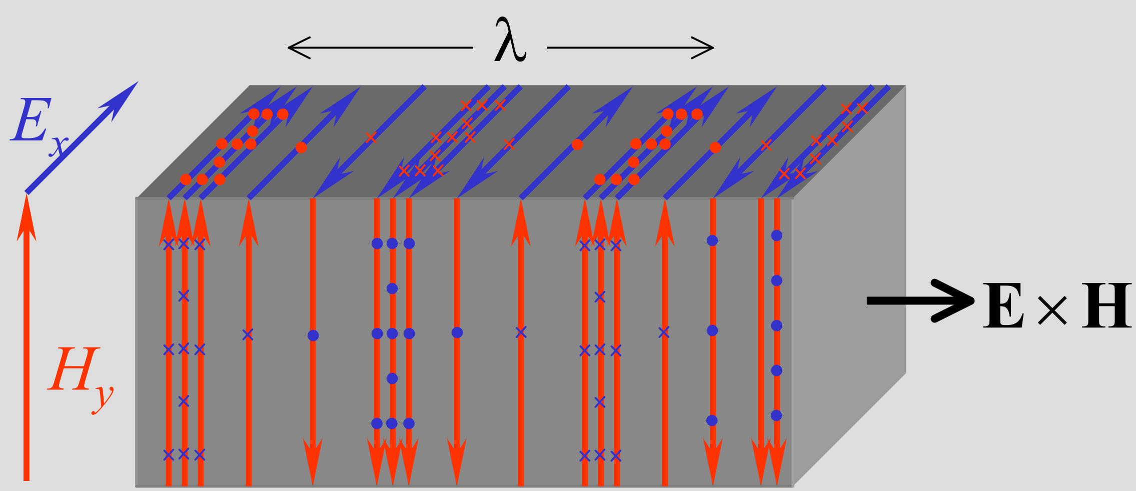
- Sinusoidal variation of E and H
- E and H in phase and orthogonal

$$H_y = \sqrt{\frac{\varepsilon}{\mu_o}} E_x$$



A block of space containing an EM plane wave

- Every point in 3D space is characterised by
  - E_x, E_y, E_z
  - Which determine
    - H_x, H_y, H_z
    - and vice versa
  - 3 degrees of freedom



$$H_y = \sqrt{\frac{\varepsilon}{\mu_o}} E_x$$

$$H_x = -\sqrt{\frac{\varepsilon}{\mu_o}} E_y$$

Power flow of EM radiation

$$u_E = \frac{\varepsilon E^2}{2}$$

$$u_H = \frac{\mu_o H^2}{2}$$

$$H_y = \sqrt{\frac{\varepsilon}{\mu_o}} E_x$$

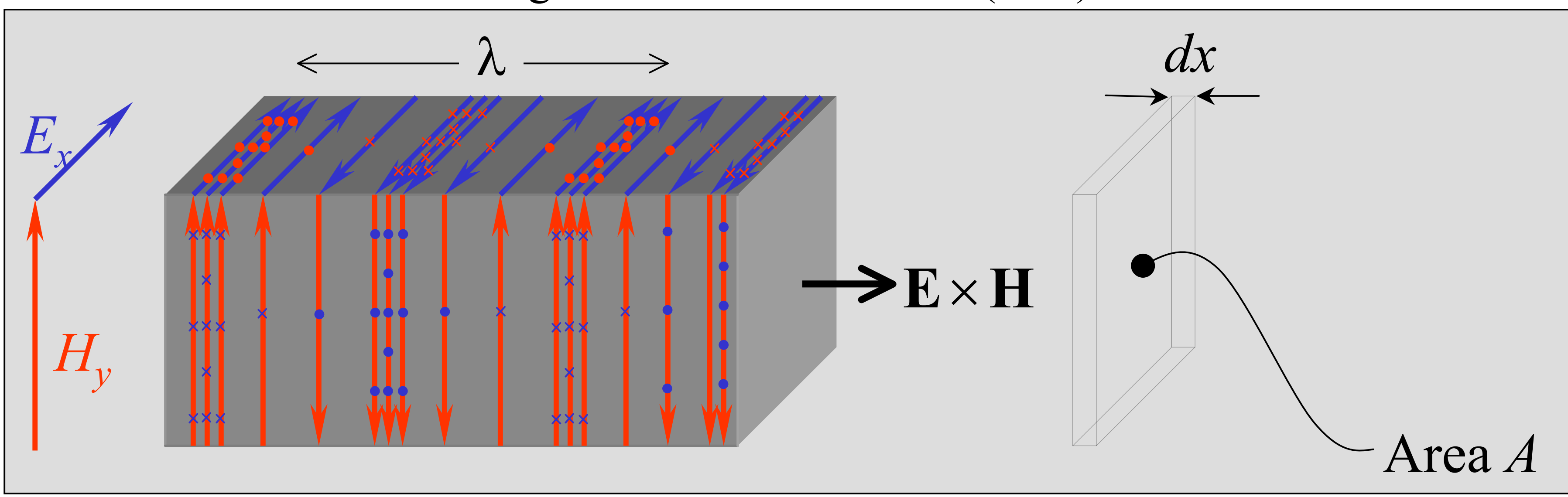
- Energy stored in the EM field in the thin box is:

$$\mathrm{d}U = \mathrm{d}U_E + \mathrm{d}U_H = \left(u_E + u_H\right) A\,\mathrm{d}x$$

$$\mathrm{d}U = \left(\frac{\varepsilon E^2}{2} + \frac{\mu_o H^2}{2}\right) A\,\mathrm{d}x$$

$$= \varepsilon E^2 A\,\mathrm{d}x$$

- Power transmitted through the box is dU/dt=dU/(dx/c)....



Power flow of EM radiation cont.

$$\mathrm{d}U = \varepsilon E^2 A\,\mathrm{d}x$$

$$S = \frac{\mathrm{d}U}{A\,\mathrm{d}t} = \frac{\varepsilon E^2}{A(\mathrm{d}x/c)} A\,\mathrm{d}x = \sqrt{\frac{\varepsilon}{\mu_o}} = \frac{E^2}{\eta} \qquad \mathrm{W/m^2}$$

- This is the instantaneous power flow
  - Half is contained in the electric component
  - Half is contained in the magnetic component
- E varies sinusoidal, so the average value of S is obtained as:

$$\mathrm{E} = E_o \sin\frac{2\pi}{\lambda}(z - vt)$$

$$S = \frac{E_o^2 \sin^2(z - vt)}{\eta}$$

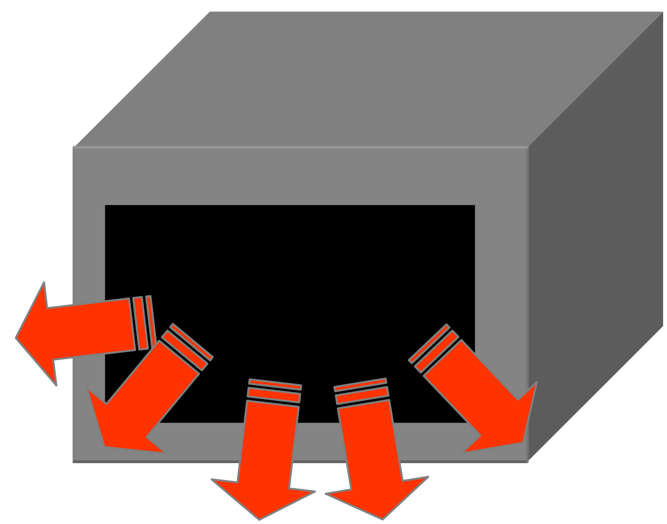
$$\overline{S} = \frac{E_o^2}{\eta}\mathrm{RMS}\left(E_o^2 \sin^2(z - vt)\right) = \frac{E_o^2}{2\eta}$$

- S is the Poynting vector and indicates the direction and magnitude of power flow in the EM field.

Example problem

- The door of a microwave oven is left open
  - estimate the peak *E* and *H* strengths in the aperture of the door.
  - Which plane contains both E and H vectors ?
  - What parameters and equations are required?
    - Power-750 W
    - Area of aperture - 0.3 x 0.2 m
    - impedance of free space - 377 Ω
    - Poynting vector:

$$S = \frac{E^2}{\eta} = \eta H^2 \quad \mathrm{W/m^2}$$



Solution

$$Power = SA = \frac{E^2}{\eta} A \quad = \eta H^2 \text{ A} \qquad \text{Watts}$$

$$E = \sqrt{\eta \frac{Power}{A}} = \sqrt{377 \frac{750}{0.3.0.2}} = 2{,}171\,\text{V/m}$$

$$\text{H} = \frac{E}{\eta} = \frac{2170}{377} = 5.75\text{A/m}$$

$$\text{B} = \mu_o H = 4\pi \times 10^{-7} \times 5.75 = 7.2\mu\text{Tesla}$$

Constitutive relations

- permittivity of free space ε_0=8.85 x 10^{-12} F/m
- permeability of free space μ_o=4πx10^{-7} H/m
- Normally ε_r (dielectric constant) and μ_r
  - vary with material
  - are frequency dependant
    - For non-magnetic materials μ_r ~1 and for Fe is ~200,000
    - ε_r is normally a few ~2.25 for glass at optical frequencies
  - are normally simple scalars (i.e. for *isotropic* materials) so that **D** and **E** are parallel and **B** and **H** are parallel
    - For ferroelectrics and ferromagnetics ε_r and μ_r depend on the relative orientation of the material and the applied field:

$$\mathbf{D} = \varepsilon\mathbf{E} = \varepsilon_r\varepsilon_o\mathbf{E}$$

$$\mathbf{B} = \mu\mathbf{H} = \mu_r\mu_o\mathbf{H}$$

$$\mathbf{J} = \sigma\mathbf{E}$$

$$\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \begin{pmatrix} \mu_{xx} & \mu_{xy} & \mu_{xz} \\ \mu_{yx} & \mu_{yy} & \mu_{yz} \\ \mu_{zx} & \mu_{zy} & \mu_{zz} \end{pmatrix} \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}$$

At microwave frequencies:

$$\mu_{ij} = \begin{pmatrix} \mu & -\mathrm{j}\kappa & 0 \\ \mathrm{j}\kappa & \mu & 0 \\ 0 & 0 & \mu_o \end{pmatrix}$$

Constitutive relations cont...

- What is the relationship between ε and refractive index for non magnetic materials ?
  - $v=c/n$ is the speed of light in a material of refractive index n

$$v = \frac{1}{\sqrt{\mu_o \varepsilon_o \varepsilon_r}} = \frac{c}{n}$$

$$n = \sqrt{\varepsilon_r}$$

  - For glass and many plastics at optical frequencies
    - n~1.5
    - ε_r~2.25
- Impedance is lower within a dielectric

$$\eta = \sqrt{\frac{\mu_o \mu_r}{\varepsilon_o \varepsilon_r}}$$

What happens at the boundary between materials of different n, μ_r, ε_r ?

Why are boundary conditions important ?

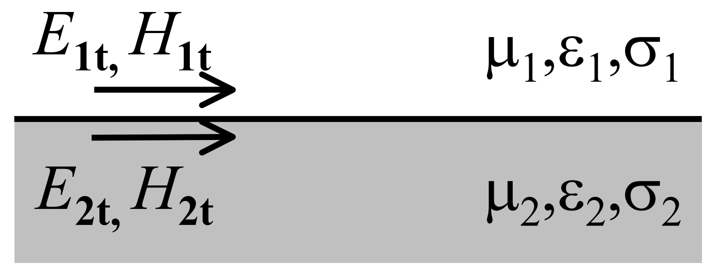
- When a free-space electromagnetic wave is incident upon a medium secondary waves are
  - transmitted wave
  - reflected wave
- The transmitted wave is due to the **E** and **H** fields at the boundary as seen from the incident side
- The reflected wave is due to the **E** and **H** fields at the boundary as seen from the transmitted side
- To calculate the transmitted and reflected fields we need to know the fields at the boundary
  - These are determined by the boundary conditions

Boundary Conditions cont.

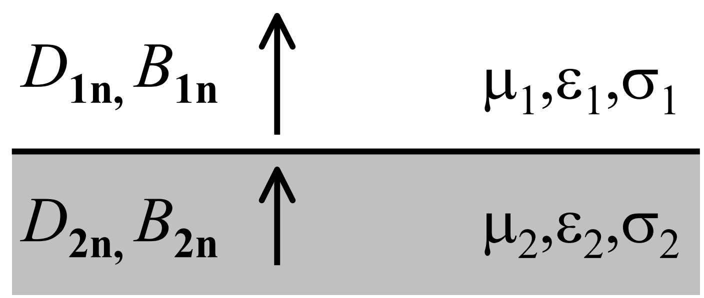
- At a boundary between two media, $\mu_r, \varepsilon_r \sigma$ are different on either side.
- An abrupt change in these values changes the characteristic impedance experienced by propagating waves
- Discontinuities results in partial reflection and transmission of EM waves
- The characteristics of the reflected and transmitted waves can be determined from a solution of Maxwells equations along the boundary

Boundary conditions

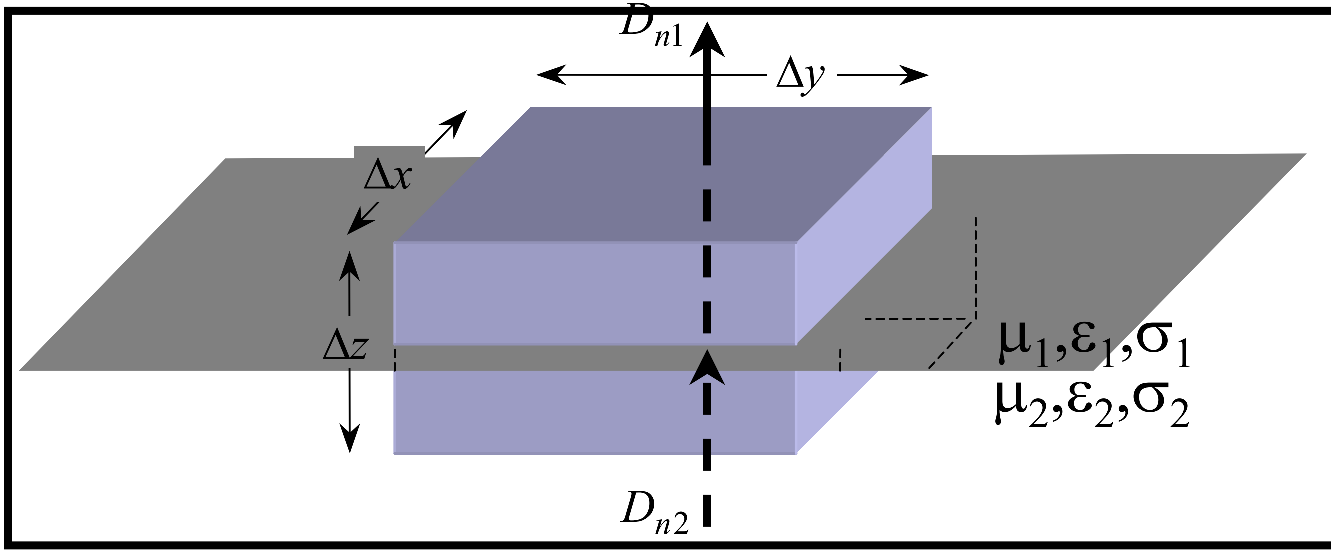
- The tangential component of **E** is continuous at a surface of discontinuity
  - $E_{\mathbf{1t},} = E_{\mathbf{2t}}$
- **Except for a perfect conductor**, the tangential component of **H** is continuous at a surface of discontinuity
  - $H_{\mathbf{1t},} = H_{\mathbf{2t}}$



- The normal component of **D** is continuous at the surface of a discontinuity if there is no surface charge density. If there is surface charge density **D** is discontinuous by an amount equal to the surface charge density.
  - $D_{\mathbf{1n},} = D_{\mathbf{2n}} + \rho_s$
- The normal component of B is continuous at the surface of discontinuity
  - $B_{\mathbf{1n},} = B_{\mathbf{2n}}$



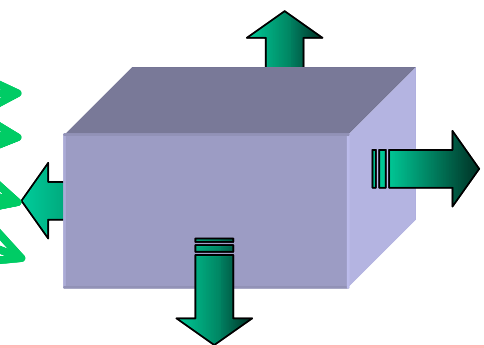
Proof of boundary conditions - $\underline{\mathbf{D}}_{\mathbf{n}}$



- The integral form of Gauss' law for electrostatics is:

$$\oiint \mathbf{D}.\mathrm{d}\mathbf{A} = \iiint_V \rho dV$$

applied to the box gives



$$D_{n1}\Delta x\Delta y - D_{n2}\Delta x\Delta y + \Psi_{\text{edge}} = \rho_s \Delta x\Delta y$$

As $\mathrm{d}z \to 0, \Psi_{\text{edge}} \to 0$ hence

$$\boxed{D_{n1} - D_{n2} = \rho_s}$$

The change in the normal component of **D** at a boundary is equal to the surface charge density

Proof of boundary conditions - $\underline{\mathbf{D}}_n$ cont.

$$D_{n1} - D_{n2} = \rho_s$$

- For an insulator with no static electric charge $\rho_s=0$

$$D_{n1} = D_{n2}$$

- For a conductor all charge flows to the surface and for an infinite, plane surface is uniformly distributed with area charge density ρ_s

  In a good conductor, σ is large, $\mathbf{D}=\varepsilon\mathbf{E}\approx 0$ hence if medium 2 is a good conductor

$$D_{n1} = \rho_s$$

Proof of boundary conditions - $\underline{\mathbf{B}}_n$

- Proof follows same argument as for D_n on page 47,
- The integral form of Gauss' law for magnetostatics is

$$\oiint \mathbf{B}.d\mathbf{A} = 0$$

  - there are no isolated magnetic poles

$$B_{n1}\Delta x\Delta y - B_{n2}\Delta x\Delta y + \Psi_{\text{edge}} = 0$$

$$\Rightarrow \qquad \boxed{B_{n1} = B_{n2}}$$

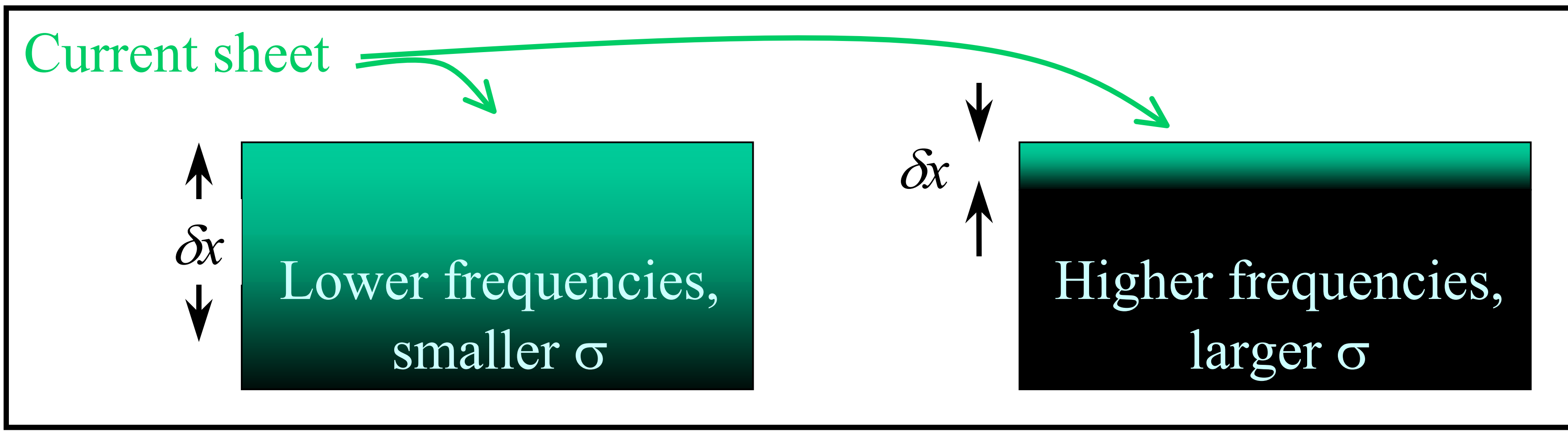
The normal component of **B** at a boundary is always continuous at a boundary

Conditions at a perfect conductor

- In a perfect conductor σ is infinite
- Practical conductors (copper, aluminium silver) have very large σ and field solutions assuming infinite σ can be accurate enough for many applications
  - Finite values of conductivity are important in calculating Ohmic loss
- For a conducting medium
  - $\mathbf{J}=\sigma\mathbf{E}$
    - infinite $\sigma \Rightarrow$ infinite **J**
    - More practically, σ is very large, **E** is very small (≈ 0) and **J** is finite

Conditions at a perfect conductor

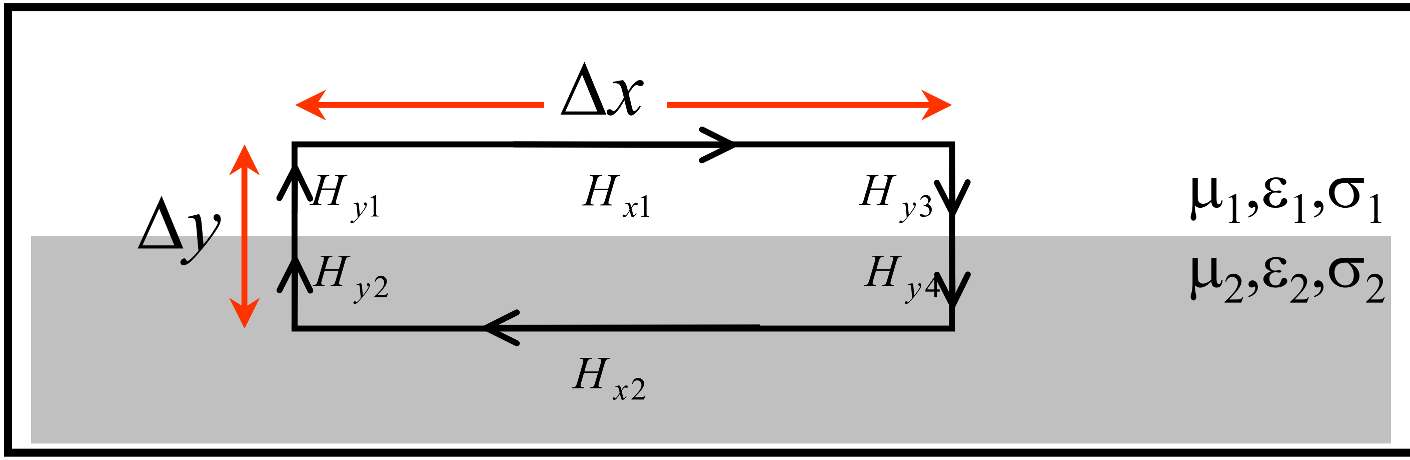
- It will be shown that at high frequencies **J** is confined to a surface layer with a depth known as the *skin depth*
- With increasing frequency and conductivity the skin depth, δx becomes thinner



- It becomes more appropriate to consider the current density in terms of current per unit with:

$$\lim_{\delta x \to 0} \mathbf{J}\,\delta x = \mathbf{J}_s \ \text{A/m}$$

Conditions at a perfect conductor cont.



- Ampere's law: $\oint \mathbf{H}.d\mathbf{s} = \iint_A \left(\frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \right) .d\mathbf{A}$

$$H_{y2} \overbrace{\frac{\Delta y}{2}}^{0} + H_{y1} \overbrace{\frac{\Delta y}{2}}^{0} + H_{x1}\Delta x - H_{y3} \underbrace{\frac{\Delta y}{2}}_{0} - H_{y4} \overbrace{\frac{\Delta y}{2}}^{0} - H_{x2}\Delta x = \overbrace{\left(\frac{\partial D_z}{\partial t} + J_z \right) \Delta x \Delta y}^{J_{sz}\Delta x}$$

$$\text{As } \Delta y \to 0, \qquad \partial D_z / \partial t \, \Delta x \Delta y \to 0, \qquad J_z \Delta x \Delta y \to \Delta x J_{sz}$$

$$\boxed{H_{x1} - H_{x2} = J_{sz}}$$

That is, the tangential component of **H** is discontinuous by an amount equal to the surface current density

Conditions at a perfect conductor cont.

- From Maxwell's equations:
  - If in a conductor E=0 then dE/dT=0
  - Since $\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{dt}$

  H_{x2}=0 (it has no time-varying component and also cannot be established from zero)

$$H_{x1} = J_{sz}$$

The current per unit width, $\mathbf{J}_s$, along the surface of a perfect conductor is equal to the magnetic field just outside the surface:

- **H** and **J** and the surface normal, **n**, are mutually perpendicular: $\mathbf{J}_s = \mathbf{n} \times \mathbf{H}$

Summary of Boundary conditions

At a boundary between non-conducting media

$$\begin{aligned} E_{t1} &= E_{t2} \\ H_{t1} &= H_{t2} \\ D_{n1} &= D_{n2} \\ B_{n1} &= B_{n2} \end{aligned} \quad \equiv \quad \begin{aligned} n\times(\mathbf{E}_1 - \mathbf{E}_2) &= 0 \\ n\times(\mathbf{H}_1 - \mathbf{H}_2) &= 0 \\ n.(\mathbf{D}_1 - \mathbf{D}_2) &= 0 \\ n.(\mathbf{B}_1 - \mathbf{B}_2) &= 0 \end{aligned}$$

At a metallic boundary (large σ)

$$\begin{aligned} n\times(\mathbf{E}_1 - \mathbf{E}_2) &= 0 \\ n\times(\mathbf{H}_1 - \mathbf{H}_2) &= 0 \\ n.(\mathbf{D}_1 - \mathbf{D}_2) &= \rho_s \\ n.(\mathbf{B}_1 - \mathbf{B}_2) &= 0 \end{aligned}$$

At a perfectly conducting boundary

$$\begin{aligned} n\times\mathbf{E}_1 &= 0 \\ n\times\mathbf{H}_1 &= \mathbf{J}_s \\ n.\mathbf{D}_1 &= \rho_s \\ n.\mathbf{B}_1 &= 0 \end{aligned}$$

Reflection and refraction of plane waves

- At a discontinuity the change in μ, ε and σ results in partial reflection and transmission of a wave
- For example, consider normal incidence:

$$\text{Incident wave} = E_i e^{j(\omega t - \beta z)}$$

$$\text{Reflected wave} = E_r e^{j(\omega t + \beta z)}$$

- Where E_r is a complex number determined by the boundary conditions

Reflection at a perfect conductor

- Tangential **E** is continuous across the boundary
- For a perfect conductor **E** just inside the surface is zero
  - *E* just outside the conductor must be zero

$$E_i + E_r = 0$$
$$\Rightarrow E_i = -E_r$$

- Amplitude of reflected wave is equal to amplitude of incident wave, but reversed in phase

Standing waves

- Resultant wave at a distance -*z* from the interface is the sum of the incident and reflected waves

$$
\begin{aligned}
E_T(z,t) &= \text{incident wave} + \text{reflected wave} \\
&= E_i e^{j(\omega t-\beta z)} + E_r e^{j(\omega t+\beta z)} \\
&= E_i\left(e^{-j\beta z} - e^{j\beta z}\right)e^{j\omega t} \\
&= -2jE_i \sin\beta z\; e^{j\omega t}
\end{aligned}
$$

$$\sin\phi = \frac{e^{j\phi} - e^{j\phi}}{2j}$$

and if E_i is chosen to be real

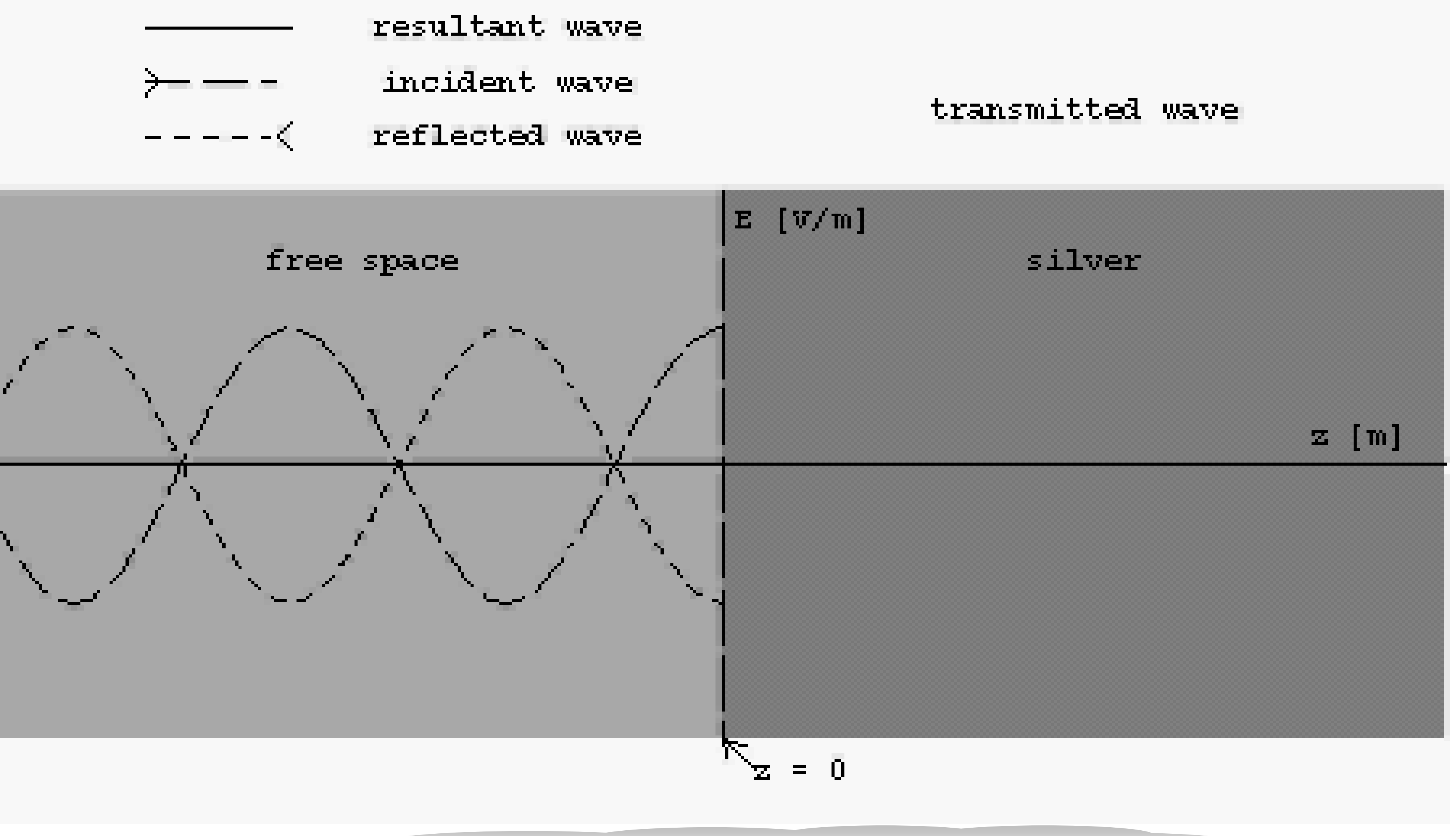
$$
\begin{aligned}
E_T(z,t) &= \operatorname{Re}\{-2jE_i \sin\beta z\left(\cos\omega t + j\sin\omega t\right)\} \\
&= 2E_i \sin\beta z \sin\omega t
\end{aligned}
$$

Standing waves cont...

$$E_T(z,t) = 2E_i \sin \beta z \sin \omega t$$

- Incident and reflected wave combine to produce a standing wave whose amplitude varies as a function (sin βz) of displacement from the interface
- Maximum amplitude is twice that of incident fields

Reflection from a perfect conductor



Reflection from a perfect conductor

- Direction of propagation is given by **E**×**H**

If the incident wave is polarised along the *y* axis:

$$E_i = \mathbf{a}_y E_{yi}$$

$$\Rightarrow H_i = -\mathbf{a}_x H_{xi}$$

then $$\mathbf{E}\times\mathbf{H} = \left(-\mathbf{a}_y \times \mathbf{a}_x\right) E_{yi} H_{xi}$$

$$= +\mathbf{a}_z E_{yi} H_{xi}$$

That is, a z-directed wave.

For the reflected wave $\mathbf{E}\times\mathbf{H} = -\mathbf{a}_z E_{yi} H_{xi}$ and $E_r = -\mathbf{a}_y E_{yi}$

So $H_r = -\mathbf{a}_x H_{xi} = H_i$ and the magnetic field is reflected without change in phase

Reflection from a perfect conductor

- Given that $\cos\phi = \dfrac{e^{j\phi} + e^{-j\phi}}{2}$

$$
\begin{aligned}
H_T(z,t) &= H_i e^{j(\omega t - \beta z)} + H_r e^{j(\omega t + \beta z)} \\
&= H_i \left(e^{j\beta z} + e^{-j\beta z}\right) e^{j\omega t} \\
&= 2H_i \cos\beta z \; e^{j\omega t}
\end{aligned}
$$

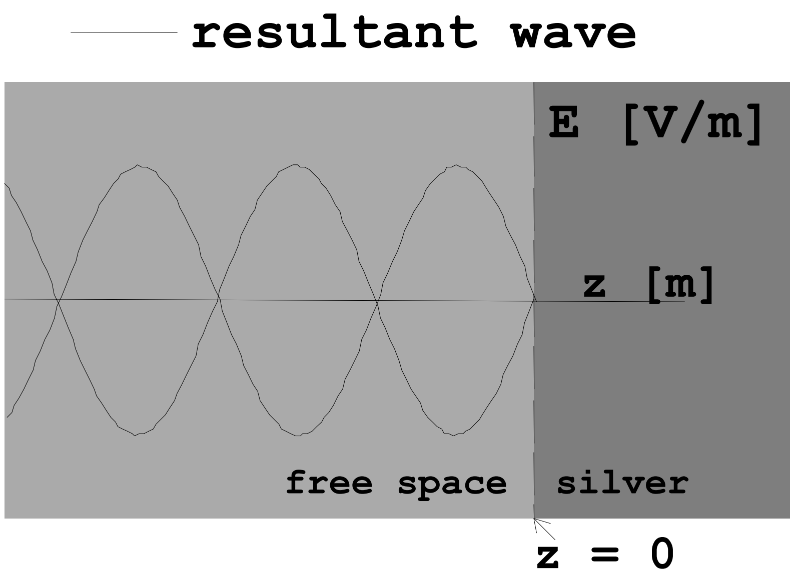
As for E_i, H_i is real (they are in phase), therefore

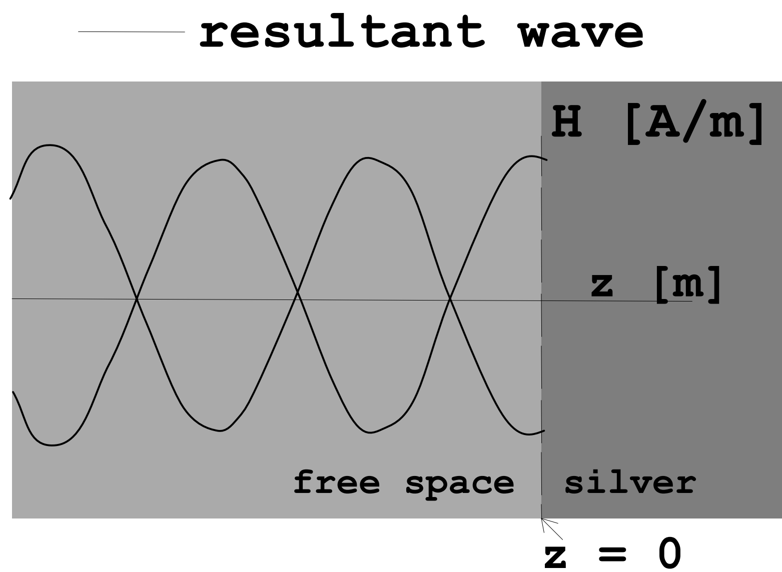
$$
H_T(z,t) = \mathrm{Re}\{2H_i \cos\beta z \; (\cos\omega t + j\sin\omega t)\} = 2H_i \cos\beta z \; \cos\omega t
$$

Reflection from a perfect conductor

$$H_T(z,t) = 2H_i \cos \beta z \cos \omega t$$

- Resultant magnetic field strength also has a standing-wave distribution
- In contrast to **E**, **H** has a maximum at the surface and zeros at $(2n+1)\lambda/4$ from the surface:





Reflection from a perfect conductor

$$E_T(z,t) = 2E_i \sin\beta z \sin\omega t$$

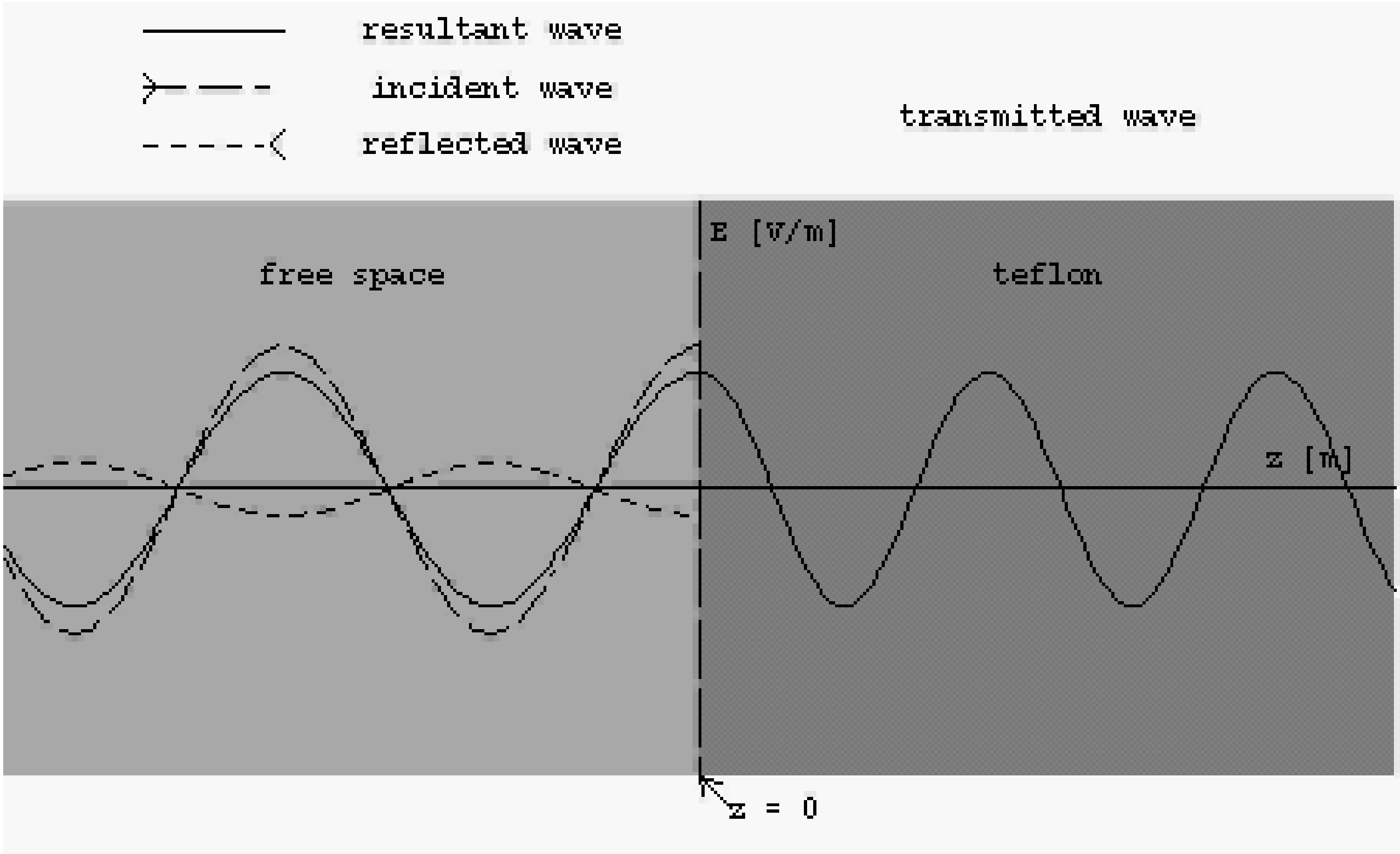
$$H_T(z,t) = 2H_i \cos\beta z \ \cos\omega t$$

- E_T and H_T are π/2 out of phase($\sin\omega t = \cos(\omega t - \pi/2)$)
- No net power flow as expected
  - power flow in +*z* direction is equal to power flow in - *z* direction

Reflection by a perfect dielectric

- Reflection by a perfect dielectric ($\mathbf{J}=\sigma\mathbf{E}=\mathbf{0}$)
  - no loss
- Wave is incident normally
  - **E** and **H** parallel to surface
- There are incident, reflected (in medium 1)and transmitted waves (in medium 2):

Reflection from a lossless dielectric



Reflection by a lossless dielectric

$$E_i = \eta_1 H_i$$
$$E_r = -\eta_1 H_r$$
$$E_t = \eta_2 H_t$$

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon_o\varepsilon_r}} = \sqrt{\frac{\mu}{\varepsilon}}$$

- Continuity of E and H at boundary requires:

$$E_i + E_r = E_t$$
$$H_i + H_r = H_t$$

Which can be combined to give

$$H_i + H_r = \frac{1}{\eta_1}\left(E_i - E_r\right) = H_t = \frac{1}{\eta_2}E_t = \frac{1}{\eta_2}\left(E_i + E_r\right)$$

$$\frac{1}{\eta_1}\left(E_i - E_r\right) = \frac{1}{\eta_2}\left(E_i + E_r\right) \quad \Rightarrow$$

$$\Rightarrow \eta_2\left(E_i - E_r\right) = \eta_1\left(E_i + E_r\right)$$

$$\Rightarrow E_i\left(\eta_2 - \eta_1\right) = E_r\left(\eta_2 + \eta_1\right)$$

$$\rho_E = \frac{E_r}{E_i} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

The reflection coefficient

Reflection by a lossless dielectric

$$E_i + E_r = E_t$$
$$H_i + H_r = H_t$$

- Similarly

$$\tau_E = \frac{E_t}{E_i} = \frac{E_r + E_i}{E_i} = \frac{E_r}{E_i} + 1 = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} + \frac{\eta_2 + \eta_1}{\eta_2 + \eta_1} = \frac{2\eta_2}{\eta_2 + \eta_1}$$

$$\tau_E = \frac{2\eta_2}{\eta_2 + \eta_1}$$

The transmission coefficient

Reflection by a lossless dielectric

- Furthermore:

$$\frac{H_r}{H_i} = -\frac{E_r}{E_i} = \rho_H$$

$$\frac{H_t}{H_i} = \frac{\eta_1 E_t}{\eta_2 E_i} = \frac{\eta_1}{\eta_2}\frac{2\eta_2}{\eta_2+\eta_1} = \frac{2\eta_1}{\eta_2+\eta_1}\tau_H$$

And because $\mu=\mu_o$ for all low-loss dielectrics

$$\rho_E = \frac{E_r}{E_i} = \frac{\sqrt{\varepsilon_1}-\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}+\sqrt{\varepsilon_2}} = \frac{n_1-n_2}{n_1+n_2} = -\rho_H$$

$$\tau_E = \frac{E_r}{E_i} = \frac{2\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_1}+\sqrt{\varepsilon_2}} = \frac{2n_1}{n_1+n_2}$$

$$\tau_H = \frac{2\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}+\sqrt{\varepsilon_2}} = \frac{2n_2}{n_1+n_2}$$

Energy Transport - Poynting Vector

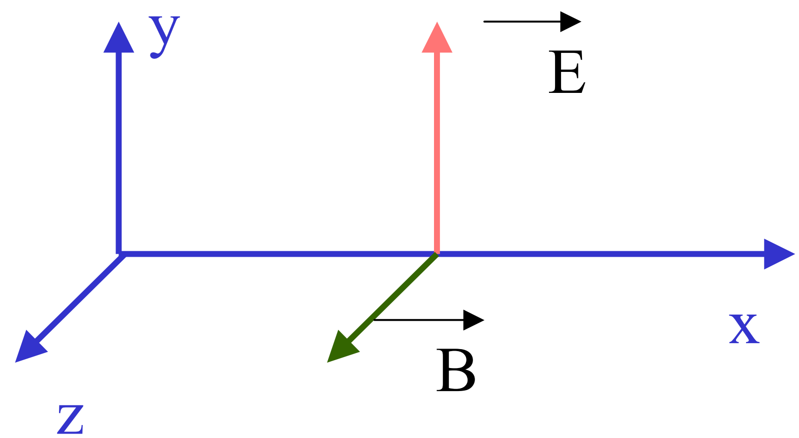
Electric and Magnetic Energy Density:

For an electromagnetic plane wave

$$\bar{E}_y(x,t) = \bar{E}_0 \sin(kx - \omega t)$$

$$\bar{B}_z(x,t) = \bar{B}_0 \sin(kx - \omega t)$$

$$\text{where } B_0 = E_0 / c$$



The electric energy density is given by

$$u_E = \frac{1}{2}\varepsilon_0 E^2 = \frac{1}{2}\varepsilon_0 \bar{E}_0^2 \sin^2(kx - \omega t)$$ *and the magnetic energy is*

$$u_B = \frac{1}{2\mu_0} B^2 = \frac{1}{2\mu_0 c} \bar{E}^2 = u_E$$

Note: I used $\bar{E} = c\bar{B}$

Energy Transport - Poynting Vector cont.

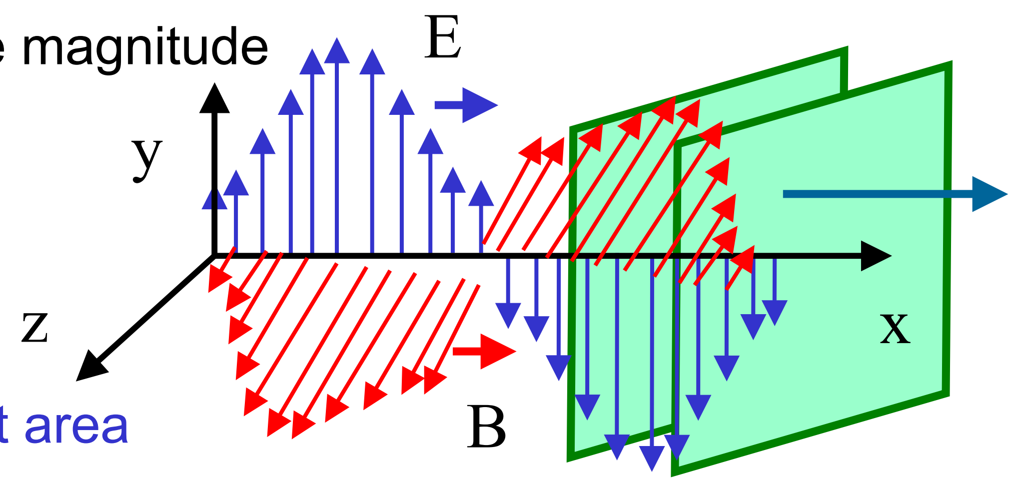
Thus, for light the electric and the magnetic field energy densities are equal and the total energy density is

$$u_{total} = u_E + u_B = \varepsilon_0 E^2 = \frac{1}{\mu_0} B^2 = \varepsilon_0 \overline{E}_0^2 \sin^2(kx - \omega t)$$

Poynting Vector $\left(\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} \right)$:

The direction of the Poynting Vector is the direction of energy flow and the magnitude

$$\left(S = \frac{1}{\mu_0} EB = \frac{E^2}{\mu_0 c} = \frac{1}{A} \frac{dU}{dt} \right)$$



Is the energy per unit time per unit area (units of Watts/m²).

Energy Transport - Poynting Vector cont.

Proof:

$$dU_{total} = u_{total}V = \varepsilon_0 E^2 Acdt \text{ so}$$

$$S = \frac{1}{A}\frac{dU}{dt} = \varepsilon_0 cE^2 = \frac{E^2}{\mu_0 c} = \frac{E_0{}^2}{\mu_0 c} sin^2(kx - \omega t)$$

Intensity of the Radiation (Watts/m^2):

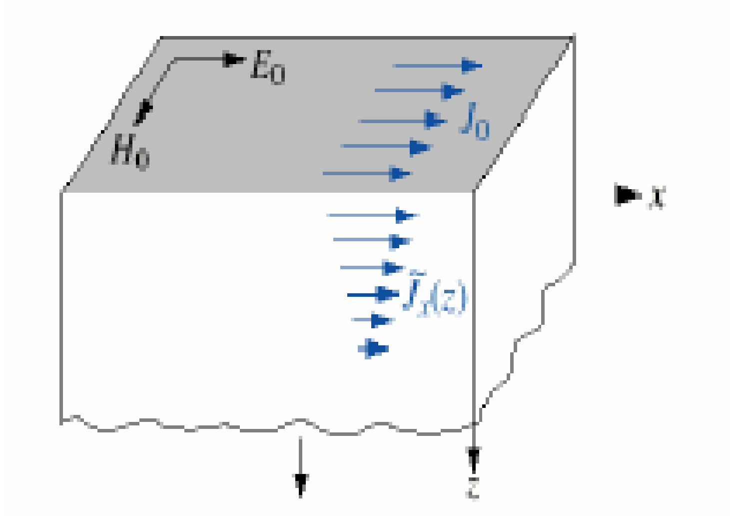
The intensity, I, is the average of S as follows:

$$I = \bar{S} = \frac{1}{A}\frac{d\overline{U}}{dt} = \frac{E_0{}^2}{\mu_0 c}\left\langle sin^2(kx - \omega t)\right\rangle = \frac{E^2}{2\mu_0 c}.$$

- Ohm's law

$$\bar{J} = \sigma \bar{E}$$

- Skin depth



Current density decays exponentially from the surface into the interior of the conductor

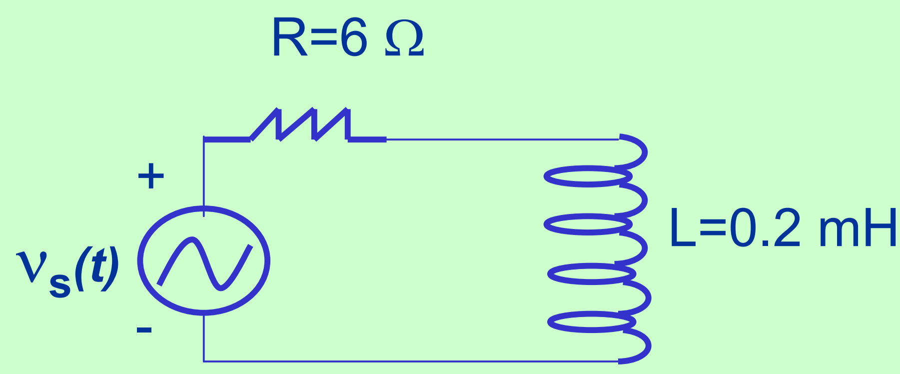
Phasors

Fictitious way of dealing with AC circuits

$$i(t) = Re\left\{ I e^{j\omega t} \right\} \qquad I = \frac{V_s}{R + j\omega L}$$

Measurable quantity

Phasor (not real)



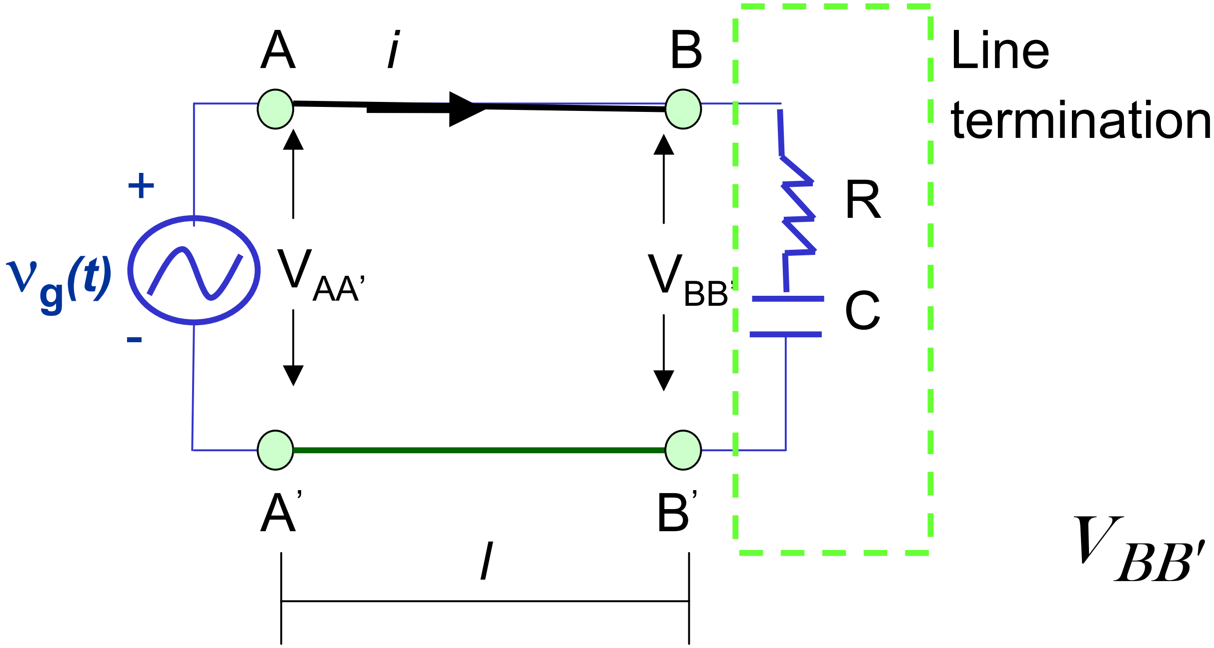
Phasors cont.

- Phasors in lumped circuit analysis have no space components

- Phasors in distributed circuit analysis (RF) have a space component because they act as waves

$$v(x,t) = Re\left\{V_0 e^{\pm j\beta x} \; e^{j\omega t}\right\} = V_0 \cos(\omega t \pm \beta x)$$

Generic Transmission Line



At line input side:

$$V_{AA'} = V_0 \cos(\omega t)$$

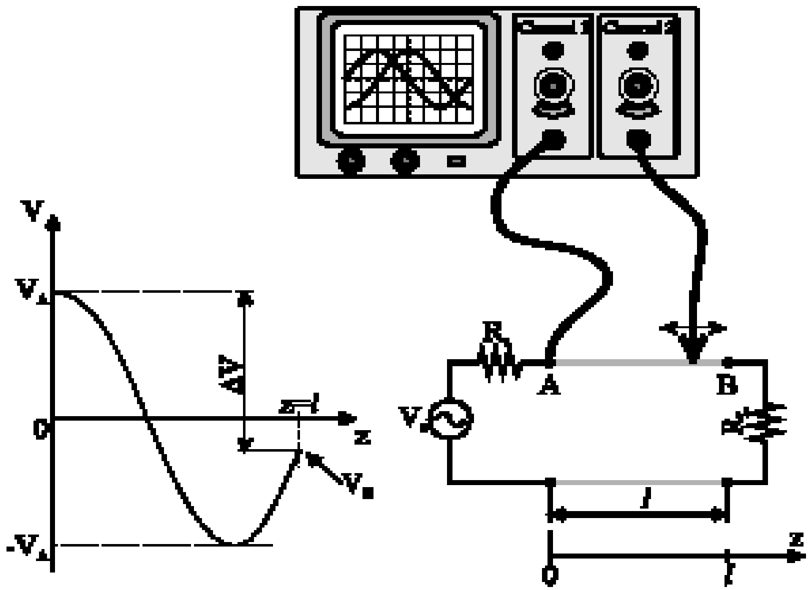
At line output side:

$$V_{BB'} = V_0 \cos\left[\omega\left(t - \frac{\ell}{c}\right)\right]$$

Is this a wave?

$$\frac{\omega}{c} = \beta \;\Rightarrow V_{BB'} = V_0 \cos[\omega t - \beta\ell]$$

Basic Measurement procedure of a transmission line



Voltage is sampled at *t = 0*

$$V_{AA'} = V_0 \, cos(0) = V_0$$

For length of 10 *cm*, *f* =1*KHz*

$$V_{BB'} = V_0 \, cos\left[\frac{2\pi \times 1kHz \times \ell}{c}\right] = 0.999...8V_0$$

For length of 10 *cm*, *f* =1*KHz*

$$V_{BB'} = V_0 \, cos\left[\frac{2\pi \times 1GHz \times \ell}{c}\right] = -0.5V_0$$

- What if low frequency, but long wire?

Frequency f = 1*kHz*, but length = 20 *km* (phone line)

$$V_{BB'} = V_0 \cos\left[\frac{2\pi \times 1kHz \times 20\,km}{c}\right] = 0.91V_0$$

- Key point: trade-off space/ frequency

$$\frac{\omega\ell}{c} = \frac{2\pi\ell}{\lambda} = 2\pi\,\frac{\ell}{\lambda}$$

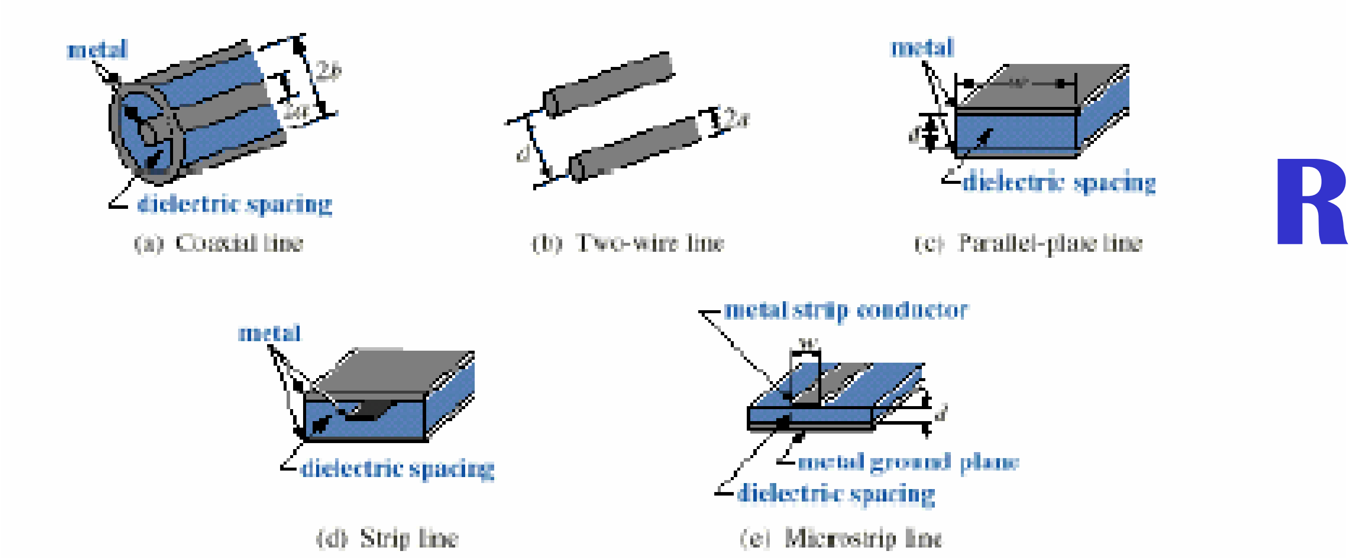
$$\frac{\ell}{\lambda} \to 0.01 \Rightarrow \cos(2\pi \times 0.01) \approx 1!$$

$$\frac{\ell}{\lambda} \geq 0.01$$ Included

$$\frac{\ell}{\lambda} \leq 0.01$$ Excluded

Trans. Effects

Types of Transmission Lines



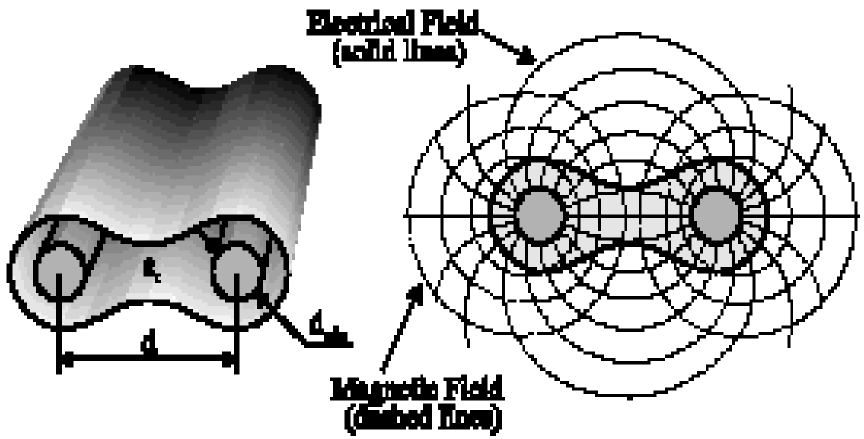
RF

TEM Transmission Lines

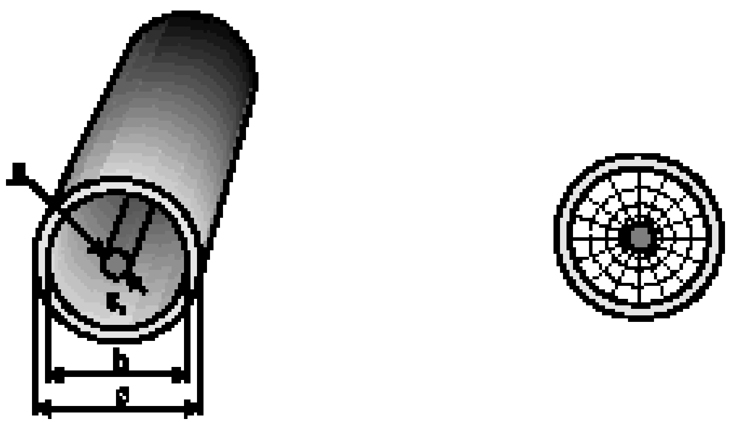
mw

Higher Order Transmission Lines

A few Transmission Line Systems



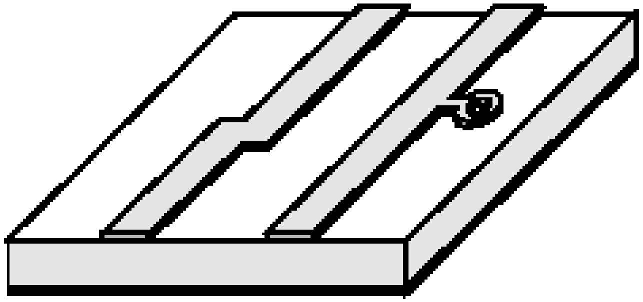
Twin-wire pair

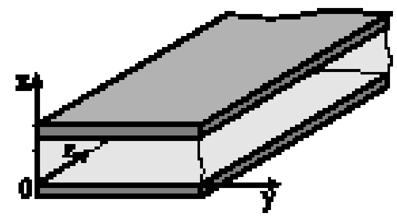


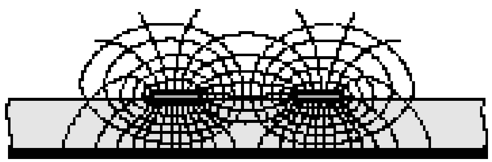
Coaxial cable

(self shielding)

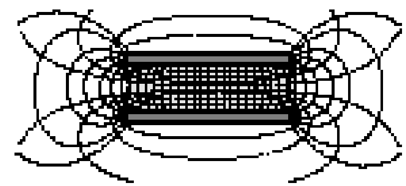
A few Transmission Line Systems (cont.)

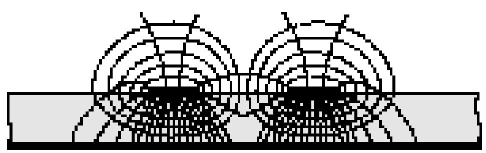






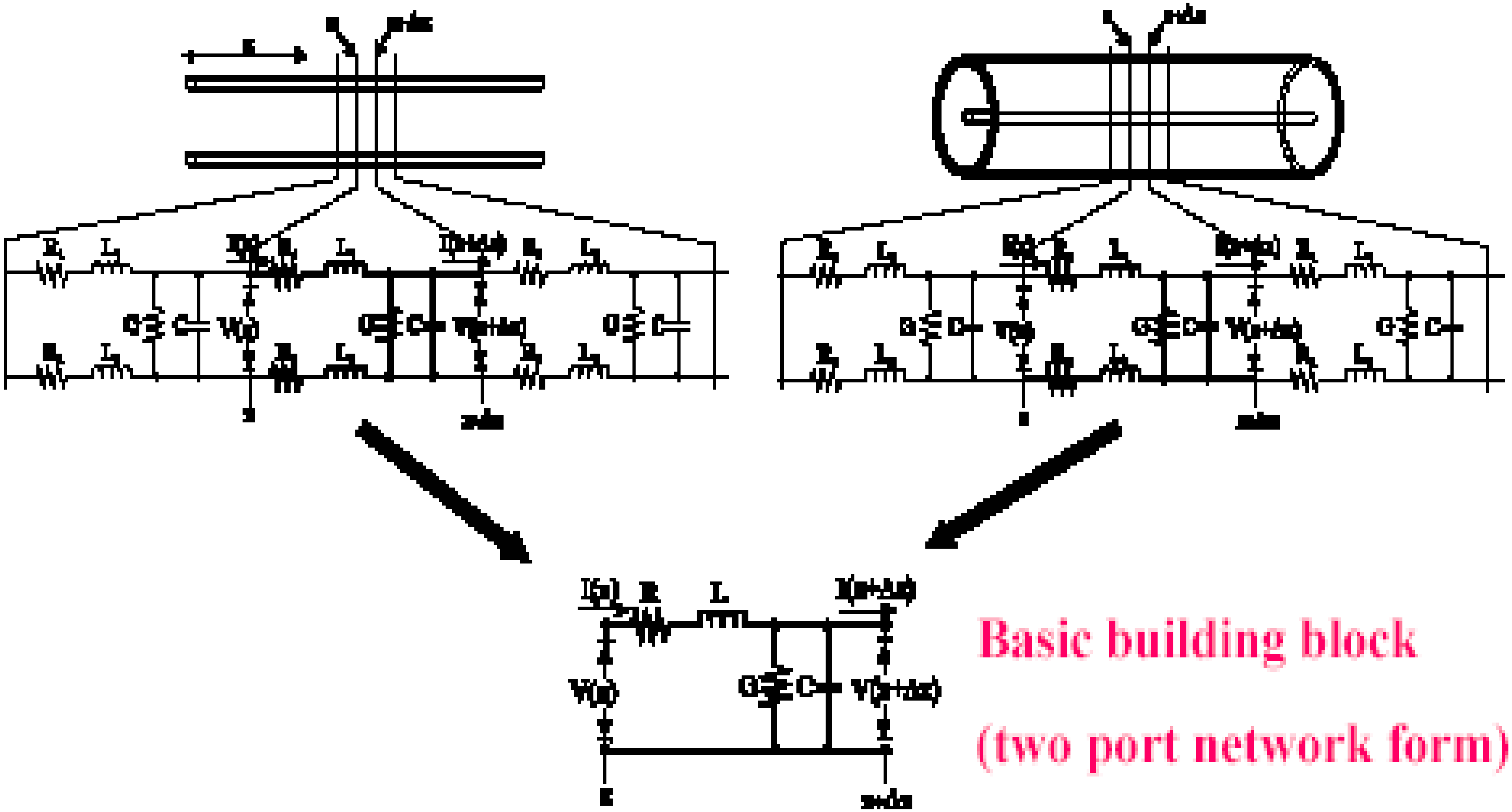
ε_r=2.55





ε_r=10.0

Common Feature of Different Transmission Lines



Traveling Voltage and Current Waves

$$\frac{d^2\tilde{V}(z)}{dz^2} - \gamma^2\tilde{V}(z) = 0$$

$$\frac{d^2\tilde{I}(z)}{dz^2} - \gamma^2\tilde{I}(z) = 0$$

$$\gamma = \alpha + j\beta = \sqrt{(R' + j\omega L')(G' + j\omega C')}$$

$$\tilde{V}(z) = V_0^+ e^{-\alpha z} e^{-j\beta z} + V_0^- e^{+\alpha z} e^{+j\beta z}$$

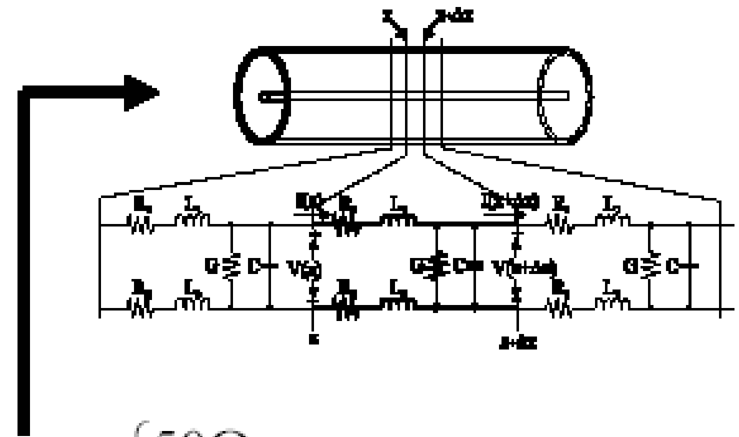
$$\tilde{I}(z) = \frac{V_0^+}{Z_0} e^{-\alpha z} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{+\alpha z} e^{+j\beta z}$$

$$Z_0 = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}}$$

Characteristic line impedance

Significance of Characteristic Line Impedance

- Independent of length
- Incorporates specific line parameters (coax,micro-strip,parallel-plate,etc.)
- Has absolutely nothing in common with the circuit element impedance



$$Z_0 = \begin{cases} 50\Omega \\ 75\Omega \end{cases}$$

- Characteristic line impedance defines wave ratio

$$\frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-}$$

- Lossless line impedance

$$Z_0 = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}} \xrightarrow{lossless} \sqrt{\frac{L'}{C'}}$$

50 ohms, RF circuits

75 ohms, antennae

Lossless Transmission Line

$$Z_0 = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}} \approx \sqrt{\frac{L'}{C'}}$$

Real characteristic line impedance

Implies

$$\alpha = 0 \text{ in } \gamma = \alpha + j\beta = \sqrt{(R' + j\omega L')(G' + j\omega C')} = j\omega\sqrt{L'C'}$$

and

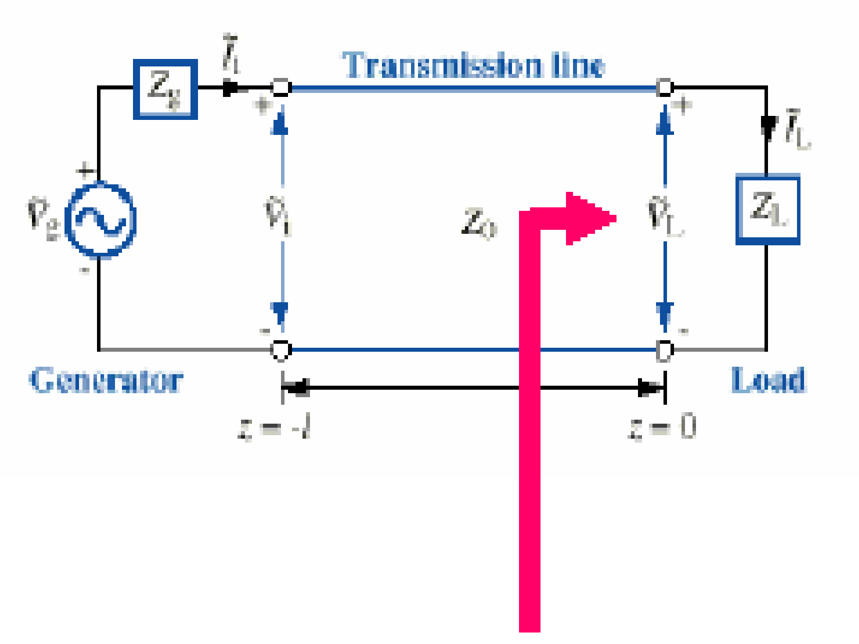
$$u_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{L'C'}}$$

Phase velocity

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{L'C'}}$$

Wave length

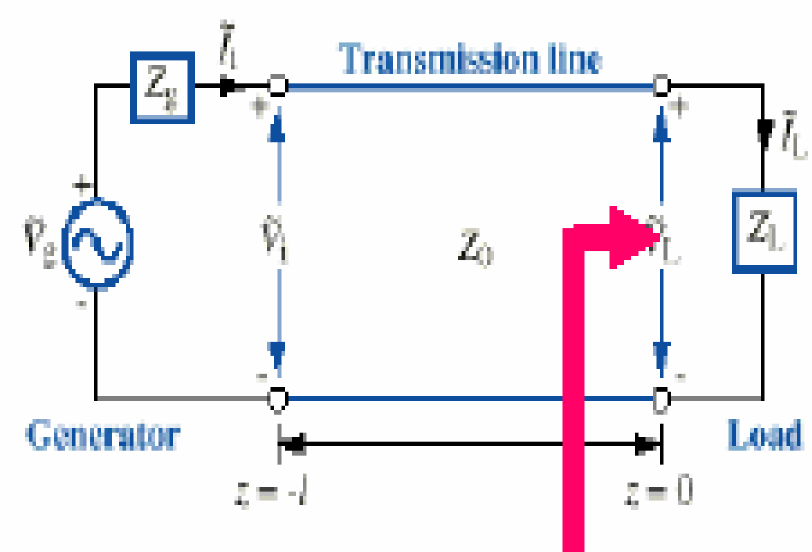
Reflection Coefficient



- Any impedance mismatch causes reflections
- Is normally a complex quantity
- Is directionally dependent (looking into the load or the source)

$$\Gamma = \frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0} = |\Gamma| e^{j\theta_r}$$

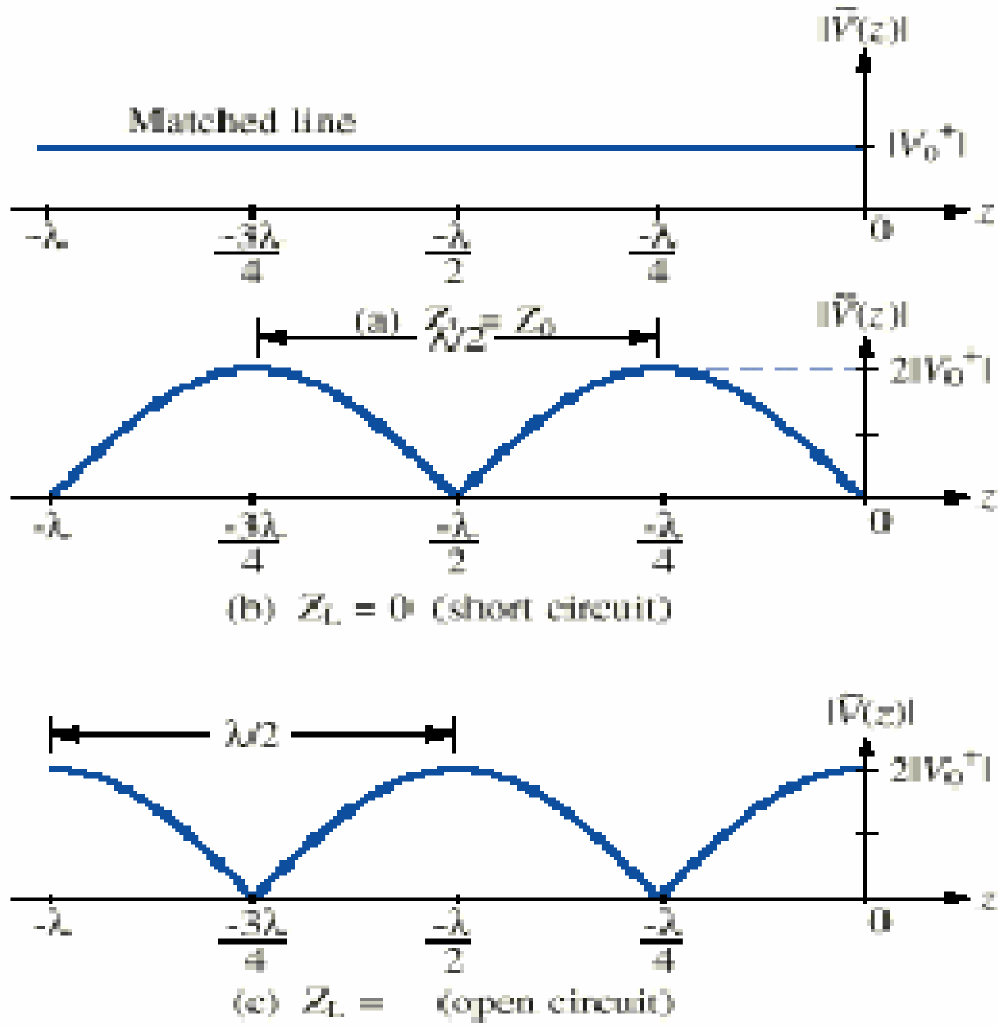
Standing Wave along a transmission line



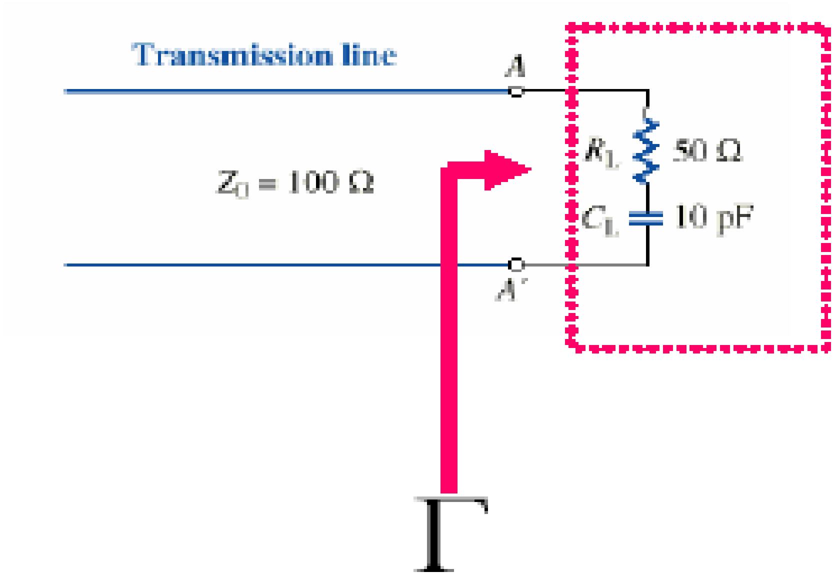
- **There are three special cases of termination:**
  - Matched line: $Z_L = Z_0 \Rightarrow \Gamma = 0$
  - Short circuit: $Z_L = 0 \Rightarrow \Gamma = -1$
  - Open circuit: $Z_L = \infty \Rightarrow \Gamma = 1$

$$\Gamma = \frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0} = |\Gamma| e^{j\theta_r}$$

Voltage behavior for the three cases:



How to quantify the amount of mismatch?



Arbitrary complex impedance

- Voltage standing wave ratio

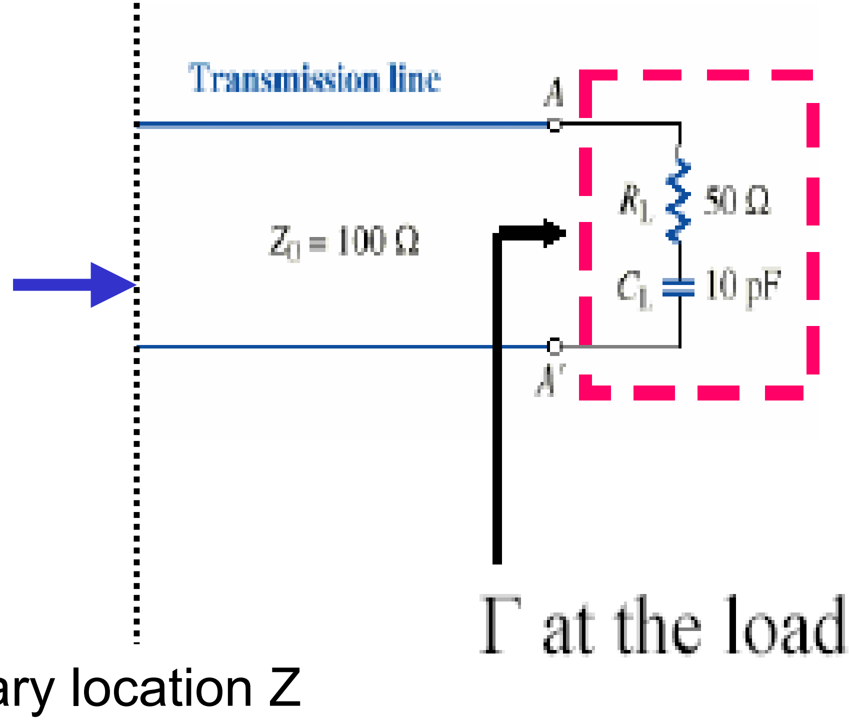
$$\mathcal{VSWR} = S = \frac{\left|\tilde{V}\right|_{max}}{\left|\tilde{V}\right|_{min}} = \frac{1+|\Gamma|}{1-|\Gamma|}$$

VSWR = 1 (matched)

VSWR = ∞ (short/open)

Input impedance of a terminated transmission line

$$Z_{in}(z) = \frac{\tilde{V}(z)}{\tilde{I}(z)}$$



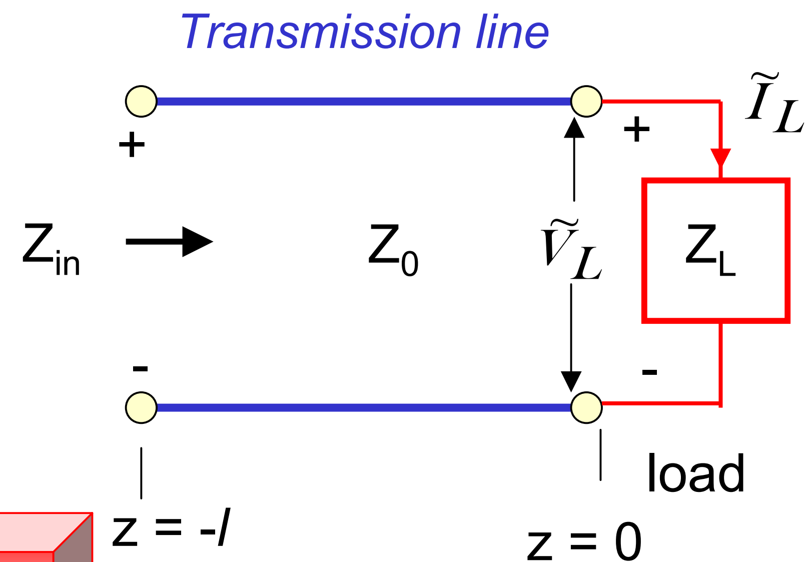
Idea is to compute the input impedance of a loaded transmission line in terms of the total voltage and current waves.

Voltage and current expressions

$$\tilde{V}(z) = V_0^+\left(e^{-j\beta z} + \Gamma e^{j\beta z}\right) \qquad \tilde{I}(z) = \frac{V_0^+}{Z_0}\left(e^{-j\beta z} - \Gamma e^{j\beta z}\right)$$

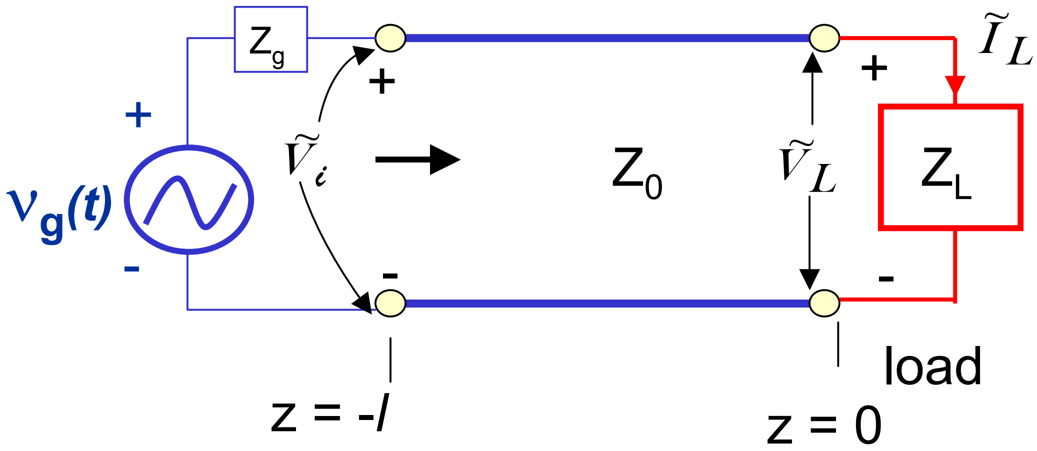
$$Z_{in}(z) = \frac{\tilde{V}(z)}{\tilde{I}(z)} = Z_0 \frac{1 + \Gamma e^{j2\beta z}}{1 - \Gamma e^{j2\beta z}}$$

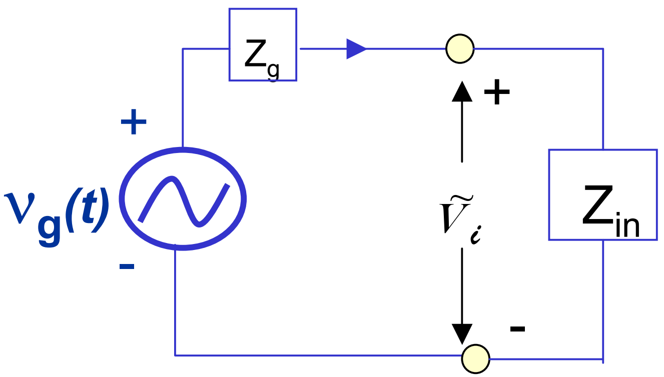
$$\boxed{Z_{in}(z) = Z_0 \frac{Z_L + jZ_0 \tan(\beta\ell)}{Z_0 + jZ_L \tan(\beta\ell)}}$$



Important T.L. equation

Including the generator into the T.L.



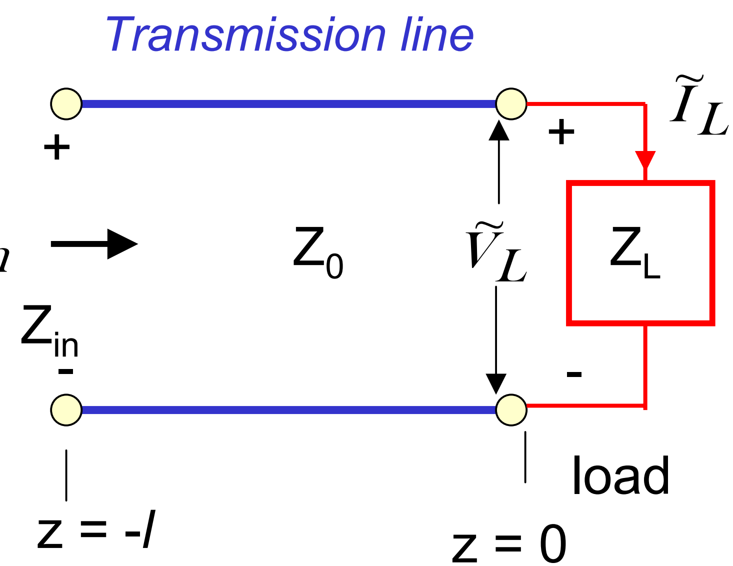


$$\tilde{V}_l = V_0^+ e^{-j\beta z}$$

$$\tilde{V}_i = \frac{\tilde{V}_g Z_{in}}{Z_g + Z_{in}} \Rightarrow V_0^+ = \frac{\tilde{V}_g Z_{in}}{Z_g + Z_{in}} \frac{1}{e^{j\beta\ell} + \Gamma e^{-j\beta\ell}}$$

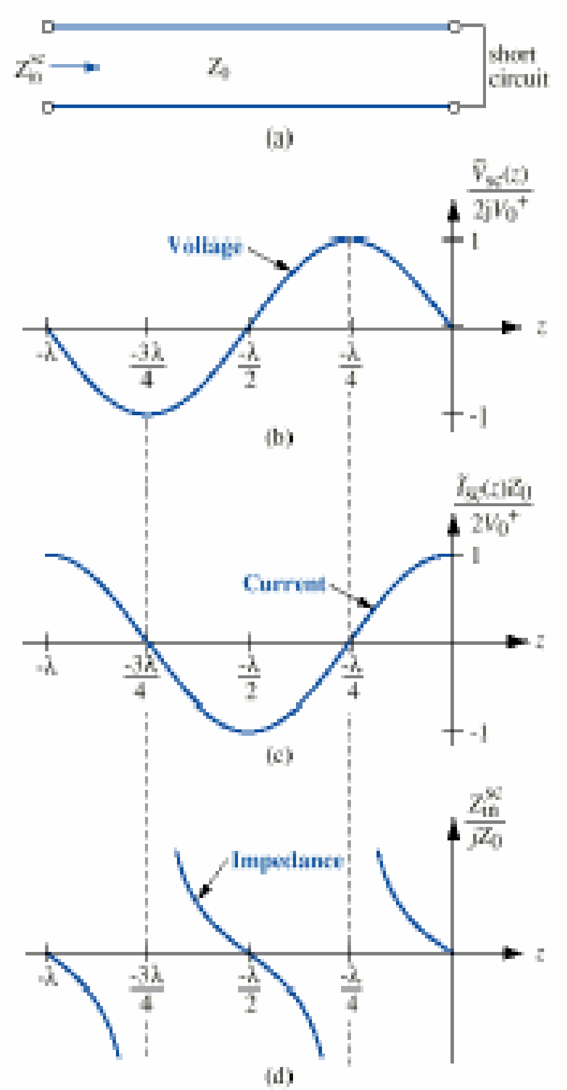
Special cases of lossless line

$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan(\beta\ell)}{Z_0 + jZ_L \tan(\beta\ell)} = R_{in} + jX_{in}$$



Input impedance can be changed almost arbitrarily depending on line length,frequency, and termination conditions.

Short circuit T.L.



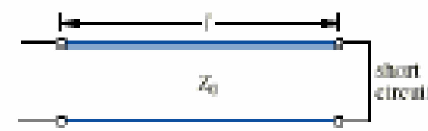
$$Z_{in}^{sc}(-\ell) = jZ_0 \tan(\beta\ell) = jX_{in}$$

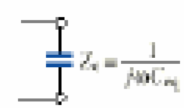
For a given inductance (L_{eq})

$$\ell = \frac{1}{\beta} \tan^{-1}\left(\frac{\omega L_{eq}}{Z_0}\right)$$

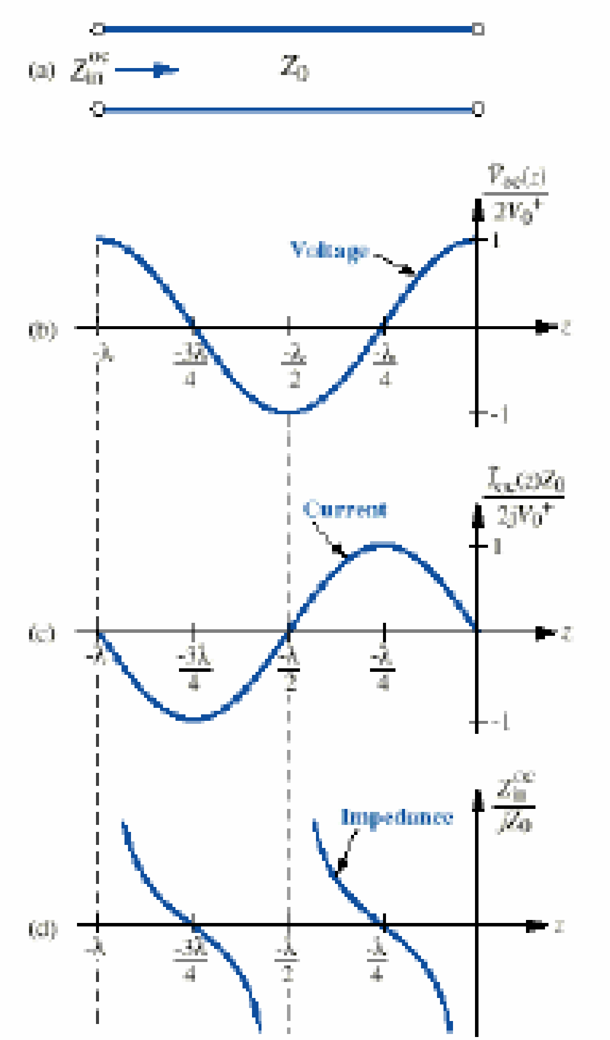
For a given capacitance (C_{eq})

$$\ell = \frac{1}{\beta}\left\{\pi - \tan^{-1}\left(\frac{1}{\omega Z_0 C_{eq}}\right)\right\}$$





Open circuit T.L.



$$Z_{in}^{os}(-\ell) = -jZ_0 \cot(\beta\ell) = jX_{in}$$

A similar procedure applies. However , an open circuit condition is difficult to enforce for high frequency opreation frequencies.

- **How to measure Characteristic line impedance and propagation constant?**

$$Z_{in}^{sc}(-\ell) = jZ_0 \tan(\beta\ell)$$

$$Z_{in}^{os}(-\ell) = -jZ_0 \cot(\beta\ell)$$

$$Z_0 = \sqrt{Z_{in}^{sc} Z_{in}^{os}}$$

$$\tan(\beta\ell) = \sqrt{\frac{-Z_{in}^{sc}}{Z_{in}^{os}}}$$

Conducting an open/short circuit measurement test with a NWA or VVM yields the characteristic line impedance.

Lambda-half line ($\ell = n\lambda/2$)

$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan(\beta\ell)}{Z_0 + jZ_0 \tan(\beta\ell)} \xrightarrow[\tan(n\pi)=0]{} Z_L$$

If the line length is multiples of $\lambda/2$, it is as if the T.L. is not present!

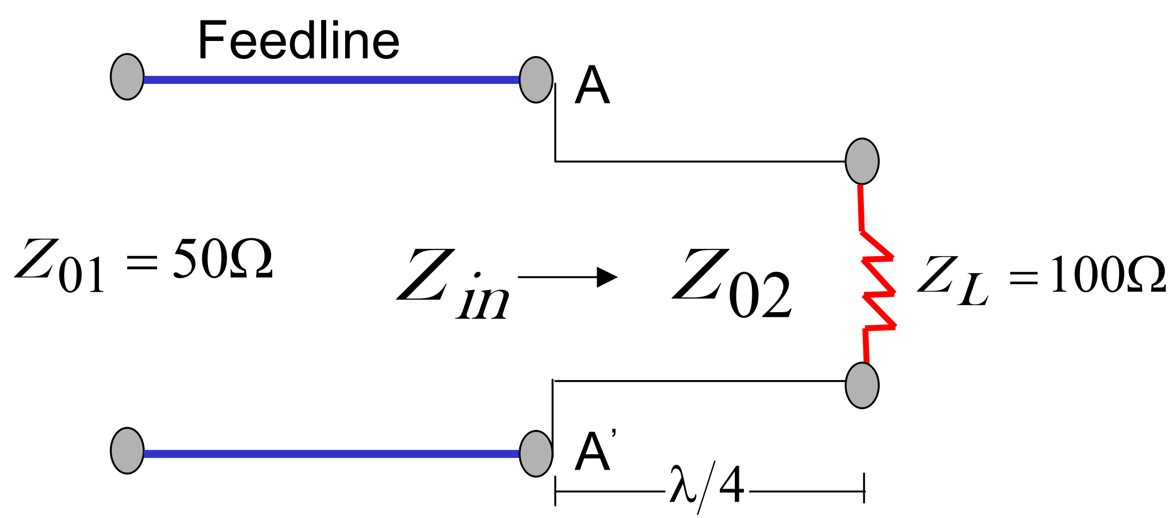
Lambda-quarter transformer

$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan(\beta \ell)}{Z_0 + jZ_0 \tan(\beta \ell)} \xrightarrow[\tan(n\pi/2) \to \infty]{} \frac{Z_0^2}{Z_L}$$

This transformation is of significant practical interest, since it allows us to match a given load impedance to a particular line impedance.

$$Z_{02} = \sqrt{Z_{01} Z_L}$$

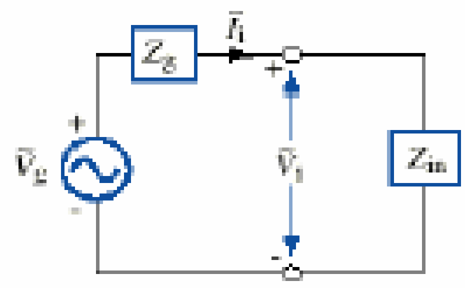
Required impedance for matching element



Power flow consideration along a lossless line

Generic average power definition

$$P_{av} = \frac{1}{2} Re\left\{\tilde{V} \cdot \tilde{I}^*\right\}$$



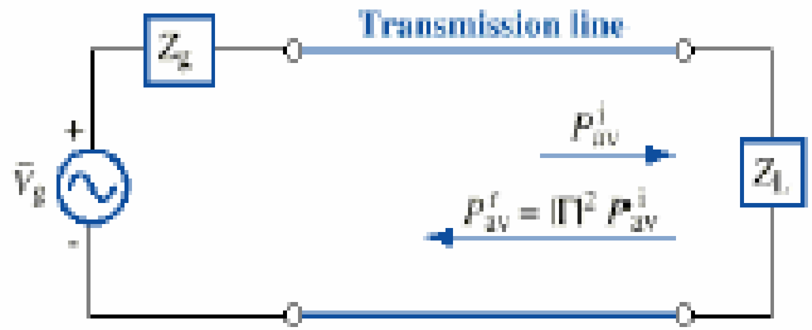
Basic power definition applies to total voltage and current expressions. For transmission lines, this means:

$$\tilde{V} = \tilde{V}_i + \tilde{V}_r$$ and $$\tilde{I} = \tilde{I}_i + \tilde{I}_r$$

Voltage/current must be split into forward and backward traveling wave components

- For a transmission line we need to modify our general power expression

$$\left.\begin{aligned}\tilde{V}_i &= V_0^+ \\ \tilde{I}_i &= \frac{V_0^+}{Z_0}\end{aligned}\right\} \quad P_{av}^i = \frac{\left|V_0^+\right|^2}{2Z_0}$$



$$P_{av} = P_{av}^i + P_{av}^r = P_{av}^i\left[1 - |\Gamma|^2\right]$$

$$\left.\begin{aligned}\tilde{V}^r &= \Gamma V_0^+ \\ \tilde{I}^r &= -\Gamma\frac{V_0^+}{Z_0}\end{aligned}\right\} \quad P_{av}^r = -|\Gamma|^2\frac{\left|V_0^+\right|^2}{2Z_0} = -|\Gamma|^2 P_{av}^i$$

Electrical properties of Materials

Classification of materials can be done by their conductivity.

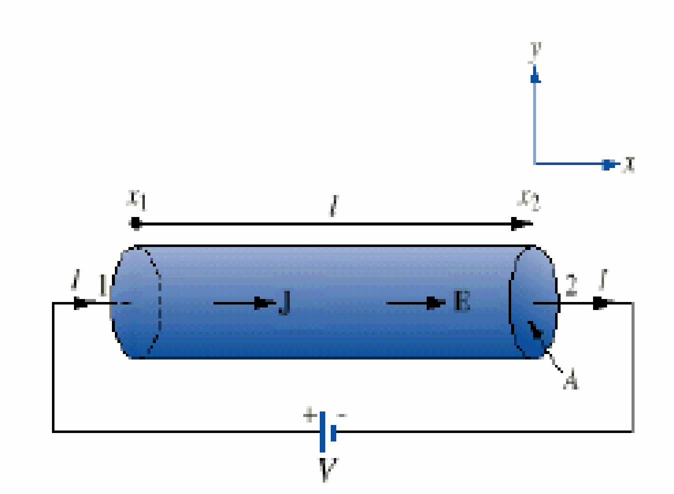
Conductors	Semi-conductors	Insulators
$\approx 10^{7}\ S\cdot m^{-1}$	$\approx 10^{-3}\, S\cdot m^{-1}$	$\approx 10^{-15}\, S\cdot m^{-1}$

$$\left(1S\cdot m^{-1}=1mho\cdot m^{-1}\right)$$

Current is composed of two charged carriers

$$J=J_e+J_h=\rho_{V_e}u_e+\rho_{V_h}u_h=\left(\rho_{V_e}\mu_e+\rho_{V_h}\mu_h\right)E=\sigma E$$

Resistance



$$R = \frac{V}{I} = \frac{l}{\sigma A}$$

Voltage drop

$$V = -\int_2^1 E \cdot d\ell = E_x \ell$$

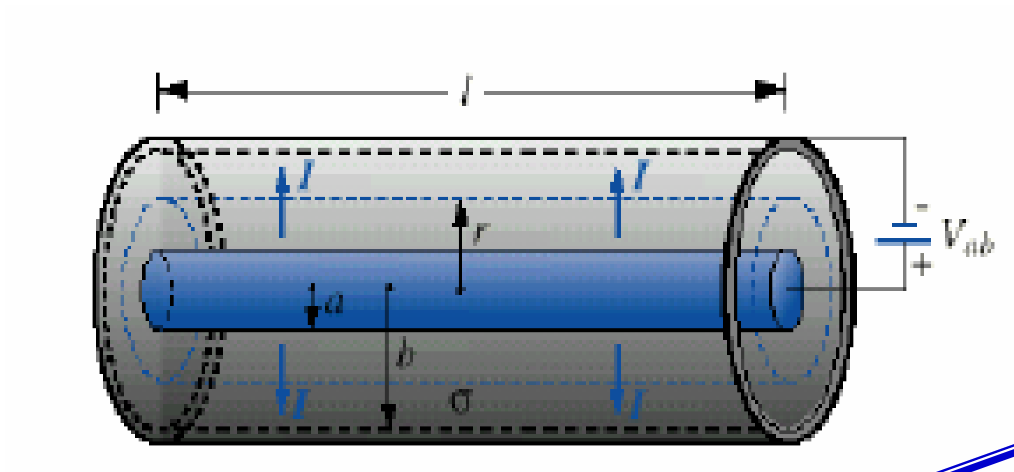
Current

$$I = -\int_2^1 J \cdot dS = \sigma E_x A$$

In general

$$R = \frac{-\int E \cdot d\ell}{\iint J \cdot dS} = \frac{-\int E \cdot d\ell}{\iint \sigma E \cdot dS}$$

Conductance of a coax-cable



$$J = \hat{r}\frac{I}{2\pi r\ell}$$

$$E = \hat{r}\frac{I}{2\pi\sigma r\ell}$$

$$V_{ab} = -\int_b^a \frac{I}{2\pi\sigma\ell}\frac{dr}{r} = \frac{I}{2\pi\sigma\ell} ln\left(\frac{b}{a}\right)$$

$$G' = \frac{G}{\ell} = \frac{2\pi\sigma}{ln\left(\frac{b}{a}\right)}$$